Recall Let $f:[a,b]\to\mathbb{R}$ be integrable. If $g:[a,b]\to\mathbb{R}$ is a funct such that $g(x)=f(x)$ for all $x\in[a,b]\setminus S$, where S is a finite, then g is integrable and $\int_a^b g=\int_a^b f$

Def A function $f:(a,b)\to\mathbb{R}$ is called integrable if there is an integrable extn $\tilde{f}:[a,b]\to\mathbb{R}$

Note: Any other extension will differ from $\tilde{f}:[a,b]\to\mathbb{R}$ by at most 2 points, so will also be integrable with the same integral

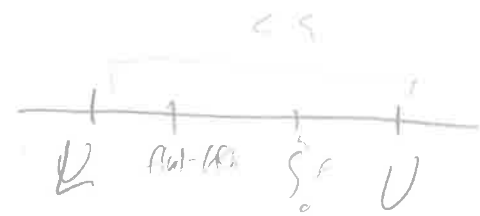
Thm Piecewise monotone and continuous functions are integrable

Proof There exists a partn $P=\{a=t_0<\dots<t_k<\dots<t_m=b\}$ such that $f:(t_{k-1},t_k)\to\mathbb{R}$ is unif cont / monotone & bounded, and the extension cont / monotone from $\tilde{f}:[t_{k-1},t_k]\to\mathbb{R}$; once $\tilde{f}$ is integrable, $f:(t_{k-1},t_k)\to\mathbb{R}$ differs from $\tilde{f}$ at at most t_{k-1},t_k, so $f:[t_{k-1},t_k]\to\mathbb{R}$ is integrable, and $f:[a,b]\to\mathbb{R}$ is integrable.

Thm (Fundamental Thm of Calculus I) (int. of a deriv.)

Let $f:[a,b]$ be cont. and diff on (a,b). If $f':(a,b)\to\mathbb{R}$ is integrable then

$$\int_a^b f' = f(b)-f(a)$$



Proof Let $\varepsilon > 0$. Since f' is int, there exists a partition P of $[a,b]$ such that

$$\boxed{U(f',P) - L(f',P) < \varepsilon.}$$

By the MVT there exists $x_k \in (t_{k-1}, t_k)$ such that

$$\frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} = f'(x_k) \text{ ; so}$$

$$\sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1}) = \sum_{k=1}^{n} f(t_k) - f(t_{k-1}) = f(b) - f(a)$$

$$f(t_1) - f(t_0) + f(t_2) - f(t_1) \dots + f(t_k) - f(t_{k-1}) \dots + f(t_n) - f(t_{n-1})$$

so $m(f', [t_{k-1}, t_k]) \le f'(x_k) \le M(f', [t_{k-1}, t_k])$

$$L(f',P) \le f(b) - f(a) \le U(f',P)$$

or $L(f',P) \le \int_a^b f \le U(f',P)$

so $\left| \int_a^b f - (f(b) - f(a)) \right| < \varepsilon$. Since this is true for all $\varepsilon > 0$, $\int_a^b f = f(b) - f(a)$.

Notation if $a > b$, $\int_a^b f = -\int_b^a f$.

Thm (Fund Thm of Calc, II) (deriv of int)
Let $f: [a,b]$ be int, define $F: [a,b] \to \mathbb{R}$, $F(x) = \int_a^x f$.

Then F is continuous; if f is continuous at $x_0 \in (a,b)$, F is diff at x_0, and $F'(x_0) = f(x_0)$.

Proof

Let $\varepsilon>0$. Choose M such that $|f(x)|<M$ for all $x\in[a,b]$, choose $\delta=\frac{\varepsilon}{M}$.

If $x,y\in[a,b]$, $x<y$ and $y-x<\delta$, then

$$|F(y)-F(x)|=\left|\int_a^y f-\int_a^x f\right|=\left|\int_x^y f\right|\le\int_x^y|f|\le\int_x^y M=M(y-x)<\varepsilon.$$

So F is continuous

Now assume f is continuous at $x_0\in(a,b)$. For $x\neq x_0$

$$\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)=\frac{1}{x-x_0}\left(\int_a^x f-\int_a^{x_0} f\right)-f(x_0)$$

$$=\frac{1}{x-x_0}\int_{x_0}^x f-\frac{1}{x-x_0}\int_{x_0}^x f(x_0)$$

$$=\frac{1}{x-x_0}\int_{x_0}^x (f-f(x_0)).$$

Let $\varepsilon>0$. Choose δ such that $x\in(a,b)\setminus\{x_0\}$, $|x-x_0|<\delta$ implies $|f(x)-f(x_0)|<\frac{\varepsilon}{2}$.

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|\le\frac{1}{|x-x_0|}\int_{x_0}^x|f-f(x_0)|\le\frac{\varepsilon}{2}<\varepsilon.$$

So $\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=f(x_0)=F'(x_0)$.

Cor (Integration by Parts) $u,v:[a,b]\to\mathbb{R}$ cont, diff on (a,b), u',v' int.

$(uv)'=u'v+uv'$ int.

$$\int_a^b u'v+\int_a^b uv'=\int_a^b (uv)'=u(b)v(b)-u(a)v(a).$$

Cor (change of variables / substitution)

Let $u : [a,b] \to \mathbb{R}$ be continuous, diff on (a,b), and $u' : (a,b) \to \mathbb{R}$ integrable.
Let $f : u([a,b]) \to \mathbb{R}$ be continuous. Then

$$\int_a^b (f \circ u)\, u' = \int_{u(a)}^{u(b)} f$$

Proof Since u is contin, $u([a,b])$ is a compact interval, $[c,d]$.
extend f to a contin function on $\mathbb{R}$ (say $f(x) = f(c)$ for $x < c$, etc.)
Define $F : \mathbb{R} \to \mathbb{R}$, $F(x) = \int_c^x f$. Then F is diff and
$F' = f$.
Define $g : [a,b] \to \mathbb{R}$ by $g(x) = F \circ u(x)$. Then g is cont,
is diff on (a,b) and $g'(x) = F'(u(x))\, u'(x) = f \circ u(x)\, u'(x)$
is int. Then

$$\int_a^b (f \circ u) u' = \int_a^b g' = g(b) - g(a) = F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} f.$$