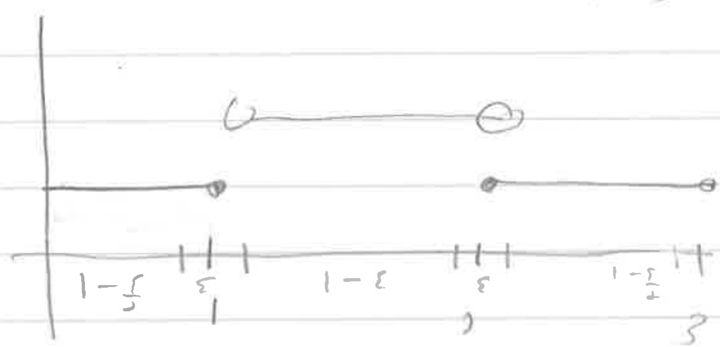
Recall A bounded fcn $f: [a,b] \to \mathbb{R}$ is integrable if and only if for all $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that
$$U(f,P) - L(f,P) < \varepsilon.$$

(Useful: Recall $L(f,P) \le \int_a^b f \le U(f,P)$)

Example 1 $f: [0,3] \to \mathbb{R}$ $f(x) = 1$ $x \in [0,1] \cup [2,3]$
$f(x) = 2$ $x \in (1,2)$



$P = \{0, 1-\frac{\delta}{2}, 1+\frac{\delta}{2}, 2-\frac{\delta}{2}, 2+\frac{\delta}{2}, 3\}$
$U(f,P) = 1(1-\frac{\delta}{2}) + 2\delta + 2(1-\delta) + 2\delta + 1(1-\frac{\delta}{2}) = 4+\delta$
$L(f,P) = 1(1-\frac{\delta}{2}) + 1\delta + 2(1-\delta) + \delta + 1(1-\frac{\delta}{2}) = 4-\delta$
$U(f,P) - L(f,P) = 2\delta$, choose $\delta < \frac{\varepsilon}{2}$, so f is int.ble
Thus $4-\delta < \int_0^3 f < 4+\delta$ for all $\delta > 0$, and $\int_0^3 f = 4$

Example 2 Let $g: [0,1] \to \mathbb{R}$, $g(x) = 1$ if $x \in \mathbb{Q}$, $g(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$
For any $[x,y]$, $x<y$, can are elements of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$, so $m(f,[x,y]) = 0$, $M(f,[x,y]) = 1$.
Thus $U(f,P) = 1$, $L(f,P) = 0$ for all partitions, so $U(f) = 1 \neq L(f) = 0$.

Properties of the Integral

Thm 1

a) If f is incr or decr, f is integrable
b) If f is contin, f is integrable

Let $f, g : [a,b] \to \mathbb{R}$ be integr

Thm 2 (Ross 33.3)

a) if $c \in \mathbb{R}$, then cf is int. and $\int_a^b cf = c\int_a^b f$
b) $f+g$ is int and $\int_a^b f+g = \int_a^b f + \int_a^b g$

Thm 3 Ross 33.4-5

a) If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$
b) $|f|$ is int. and $\left|\int_a^b f\right| \le \int_a^b |f|$
c) If f is contin & $f(x) \ge 0$ for all $x \in [a,b]$, and $\int_a^b f = 0$
then $f(x) = 0$ for all $x \in [a,b]$

Thm 4 If $f : [a,b] \to \mathbb{R}$, $f : [b,c] \to \mathbb{R}$ are int., then Ross 33.6
$f : [a,c] \to \mathbb{R}$ is int. and $\int_a^c f = \int_a^b f + \int_b^c f$.

1a) Let P have intervals of length δ

$U(f,P) = \sum f(t_k)\delta = \delta(f(t_1) + f(t_2) + \dots f(b))$

$L(f,P) = \sum f(t_{k-1})\delta = \delta(f(a) + f(t_1) + \dots f(t_{n-1}))$

$U(f,P) - L(f,P) = \delta(f(b) - f(a))$. Choose $\delta = \frac{\varepsilon}{f(b)-f(a)}$

1b) Let $\varepsilon > 0$. f is unif. cont.; choose $\delta > 0$ s.t. $|x-y| < \delta$, $\Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{b-a}$

Choose a P with intervals of length $< \delta$. i.e. each $x_i, x_0 \in [t_{k-1}, t_k]$

satisfies $M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \leq f(x_i) - f(x_0) < \frac{\varepsilon}{b-a}$

$\Rightarrow U(f,P) - L(f,P) < \sum \frac{\varepsilon}{b-a}(t_k - t_{k-1}) = \varepsilon$.

3c) Suppose $f(x_0) > 0$ for $x_0 \in (a,b)$. Then $\exists\ \delta > 0$ s.t. $f(x) > \frac{f(x_0)}{2}$ for $x \in (x_0-\delta, x_0+\delta)$. Then

$$\int_a^b f \geq \int_{x_0-\delta}^{x_0+\delta} f \geq \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2} = \delta f(x_0) > 0.$$

Thm Let $f: [a,b] \to \mathbb{R}$ be integrable. If $g: [a,b] \to \mathbb{R}$ is a function such that $g(x) = f(x)$ for all $x \in [a,b] \setminus S$, where S is finite, then g is integrable and $\int_a^b g = \int_a^b f$.

Def A bounded function

1) $f:[a,b]\to\mathbb{R}$ is piecewise monotone if there exists a partition $P=\{a=t_0<\dots<t_k<\dots<t_n=b\}$ such that $f:(t_{k-1},t_k)\to\mathbb{R}$ is monotone.

2) $f:[a,b]\to\mathbb{R}$ is piecewise continuous if there exists a partition $P=\{a=t_0<\dots<t_k<\dots<t_n=b\}$ such that $f:(t_{k-1},t_k)\to\mathbb{R}$ is uniformly continuous.