<u>Recall</u> $f: S \to \mathbb{R}$ is <u>uniformly</u> <u>continuous</u> if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$x, y \in S$, $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

<u>Thm</u> If $f: S \to \mathbb{R}$ is unif. continuous and (s_n) is a Cauchy sequence in S, $(f(s_n))$ is Cauchy.

<u>Proof</u> Let $\varepsilon > 0$. There exists $\delta > 0$ such that $x, y \in S$ and $|x-y| < \delta$ imply $|f(x)-f(y)| < \varepsilon$. There exists $N \in \mathbb{R}$ such that $n, m > N$ imply $|s_n - s_m| < \delta$. Then $n, m > N$ imply $|f(s_n) - f(s_m)| < \varepsilon$. So $(f(s_n))$ is Cauchy.

(<u>Note</u>: If S is closed, $s_n \to s \in S$, so $f(s_n) \to f(s)$ by <u>continuity</u> and is Cauchy.) But

$\left(\frac{1}{n}\right)$ in $(0,1)$ is Cauchy, $f(x) = \frac{1}{x}$, $f(\frac{1}{n}) = n$ is not Cauchy

as it converges to $0 \notin S$.

<u>Thm</u> $f: (a,b) \to \mathbb{R}$ is unif. contin. if and only if it extends to a continuous function $f: [a,b] \to \mathbb{R}$. (i.e. we can choose values of $f(a)$, $f(b)$ so that f is contin.)

i.e. $\lim_{x \to a} f(x)$, $\lim_{x \to b} f(x)$ exist $\in \mathbb{R}$

<u>Proof sketch</u> One direc. is easy: if f extends, $f: [a,b] \to \mathbb{R}$ is unif. contin. ($[a,b]$ is c.) so $f: (a,b) \to \mathbb{R}$ is unif. cont.

If $f: (a,b) \to \mathbb{R}$ is unif. cont., let (s_n) be a seq. in (a,b) conv. to b. Then (s_n) is Cauchy, so $f(s_n)$ is Cauchy, and conv. to some $y \in \mathbb{R}$. If (t_n) is another seq. conv. to b, $(s_1, t_1, s_2, t_2, \ldots)$ is Cauchy, and $(f(s_1), f(t_1), f(s_2), f(t_2), \ldots)$ is Cauchy and conv., so $f(s_n) \to y$ and $f(t_n) \to y$. Then $\lim_{x \to b} f(x) = y$.

Continuity in general metric spaces

Let S be a metric space with distance fn d
S^* be a metric space with dist fn d^*

Let $E \subseteq S$. A fn $f: E \to S^*$ gives an elmt $f(x) \in S^*$ for every $x \in E$.

Def A fn $f: E \to S^*$ is continuous at $x_0 \in E$ if
1) for all sequences (s_n) in E, $s_n \to x_0$ implies $f(s_n) \to f(x_0)$

OR

2) for all $\varepsilon > 0$ there exists $\delta > 0$ such that
$x \in E$, $d(x, x_0) < \delta$ implies $d^*(f(x), f(x_0)) < \varepsilon$
(i.e. $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$)

Def $f: E \to S^*$ is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in E$, $d(x,y) < \delta$ implies $d^*(f(x), f(y)) < \varepsilon$.

Thm If E is s.c., $f: E \to S^*$ cont, then f is unif. cont.

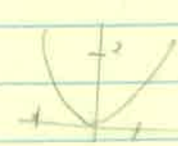
Proof: Identical to $\mathbb{R}$ proof.

Cor If $E \subset S$ is s.c. $f: E \to \mathbb{R}$ cont, then exists $x_0, x_1 \in E$ s.t. $f(x_0) \le f(x) \le f(x_1)$ for all $x \in E$.

Proof: identical.

Def "inverse image" "preimage" $f: E \to S^*$, $F \subseteq S^*$,
Confusing notation $\rightarrow$ $f^{-1}(F) = \{x \in E \mid f(x) \in F\}$

Ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$



$f^{-1}((0,2)) = (-\sqrt{2}, \sqrt{2})$
$f^{-1}((0,\infty)) = \mathbb{R}$
$f^{-1}((-1,-1)) = \emptyset$

Note $f^{-1}(S^* \setminus F) = (S) \setminus f^{-1}(F)$
$\{x \in S \mid f(x) \in S^* \setminus F\} = \{x \in S \mid f(x) \notin F\} = \{x \in S \mid x \notin f^{-1}(F)\}$