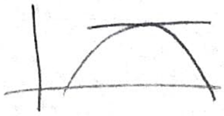


Thm Let $I \subseteq \mathbb{R}$ be an open interval and $f: I \to \mathbb{R}$ diff. at $x_0 \in I$. If f achieves its max or min at x_0 then $f'(x_0) = 0$.

L1

$f(x_0) = \max f(I)$, $f(x_0) \geq f(x)$ all $x \in I$

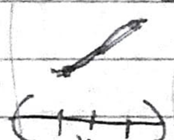
$f(x_0) = \min f(I)$, $f(x_0) \leq f(x)$ all $x \in I$.

Proof Assume $f: I \to \mathbb{R}$ is diff at x_0 and $f'(x_0) > 0$. Set $\varepsilon = f'(x_0)$. There exists $\delta > 0$ such that

L2

$x \in I$, $|x - x_0| < \delta$ implies $\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < f'(x_0)$

and thus $\frac{f(x) - f(x_0)}{x - x_0} > f'(x_0) - f'(x_0) = 0$.



Since $I \cap (x_0 - \delta, x_0 + \delta)$ is open, we can pick $a, b \in I$, $|b - x_0| < \delta$ and $|a - x_0| < \delta$ such that $a < x_0 < b$.

Thus $\frac{f(b) - f(x_0)}{b - x_0} > 0$ so $f(b) > f(x_0)$

$\frac{f(a) - f(x_0)}{a - x_0} > 0$ so $f(a) < f(x_0)$.

U2

Thus $f(x_0)$ is not the min or max of $f(I)$.

If $f'(x_0) < 0$, chose $\varepsilon = |f'(x_0)|$, show $\frac{f(x) - f(x_0)}{x - x_0} < 0$, and

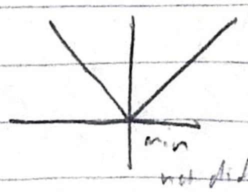
conclude similarly that $f(x_0) \neq \max, \min f(I)$.

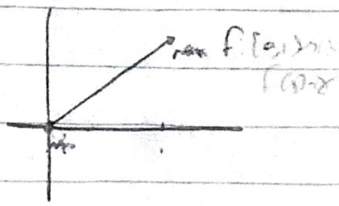
U1

when the conclu doesn't hold

$f: [-1,1] \to \mathbb{R}$

$f(x) = |x|$

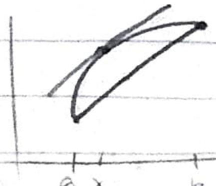




Mean Value Theorem Let $f: [a,b]$ be continuous, and diff on (a,b). Then there exists $x \in (a,b)$ such that

L1

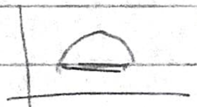
$f'(x) = \frac{f(b) - f(a)}{b - a}$



Relate Derivative to diff quotient of far apart points

diff quotient of very close points $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ pairs $\frac{f(b)-f(a)}{b-a}$

At the critical point of f where x_0 is max/min, $f'(x_0)$ ... for all $x \in [a,b]$.



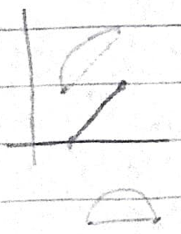
Lemma (Rolle's thm) Let $f:[a,b]$ be continuous and diff. on (a,b). If $f(a)=f(b)$ then there exists $x \in (a,b)$ such that $f'(x)=0$.

Proof: Since f is cont, $[a,b]$ compact, By the extreme value Thm there exist $x_0, y_0 \in [a,b]$ such that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in [a,b]$.
If x_0, y_0 are both endpoints (a, b) then $f(x_0)=f(y_0)$, so f is constant, and $f'(x)=0$ for all $x \in (a,b)$

Otherwise, f achieves its min or max in (a,b), so $f'=0$ at that point.

Proof (Mean Value Thm)

Let $g(x) = f(x) - (x-a)\frac{f(b)-f(a)}{b-a}$



Then g is cont, and diff. on (a,b)

$g(a) = f(a)$ $g(b) = f(b) - f(b) + f(a) = f(a)$

By Rolle's, there exists $x \in (a,b)$ such that $0 = g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

Note: If I is an open interval, $f: I \to \mathbb{R}$ diff, then $f:[a,b] \to \mathbb{R}$ satisfies the conditions of the IVT for any $a, b \in I$

(diff on (a,b), cont at a and b)

Cor 1 Let I be an open interval, $f: I \to \mathbb{R}$ diff. If f' is bounded then f is uniformly continuous

L_L

(we saw at a point, derivs let us choose δ in ε, now we apply that to make a choice that works everywhere)

Proof Suppose $|f'(x)| \leq M$ for all $x \in I$.
Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{M}$. If $a, b \in I$, $|a-b| < \delta$,

L_R

We can apply the MVT to $f: [a,b] \to \mathbb{R}$ and conclude that

$$\left| \frac{f(a) - f(b)}{a-b} \right| = |f'(x)| \leq M \quad \text{for some } x \in I,$$

so $|f(a) - f(b)| \leq M|a-b| < M\delta = \varepsilon$.

Cor 2 Let I be an open interval, $f: I \to \mathbb{R}$ diff. Let $a, b \in I$, $a < b$

If $f'(x) > 0$ for all $x \in I$	then $f(a) < f(b)$	strictly increasing
$\geq$	$\geq$	increasing
$<$	$>$	strictly decreasing
$\leq$	$\leq$	decreasing
$=$	$=$	constant

U_L

Proof There's an $x \in (a,b)$ such that $f'(x) = \dfrac{f(b) - f(a)}{b-a}$

U_R

Cor 3 (Intermediate value theorem for derivatives) Let I be an open interval, $f: I \to \mathbb{R}$ diff. Then $f'(I)$ is an interval
(i.e. if c is between $f'(a)$ and $f'(b)$, there's $x \in I$ such that $f'(x) = c$)

Note This would follow from the IVT if f' were continuous. That is not always the case; see HW 8. IVT prop holds anyway.