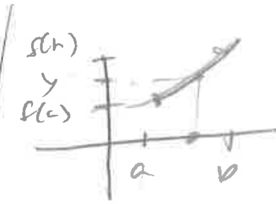
$f(S) = \{ f(x) \mid x \in S \}$ (closed + bounded)



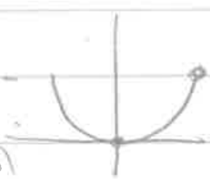
Recall If $S \subseteq \mathbb{R}$ is sequentially compact, $f: S \to \mathbb{R}$ continuous, then $f(S)$ is sequentially compact

(Note: our proof would work for any metric space)

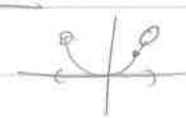
Cor Let S be S.C. and $f: S \to \mathbb{R}$ contin. There exist $x_0, x_1 \in S$ such that $f(x_0) \le f(x) \le f(x_1)$ for all $x \in S$ (i.e. f obtains its max and min value)

Proof $f(S)$ is S.C., so closed and bounded. Thus $\sup S, \inf S$ exist and are in $f(S)$. (HW 4 #5)
(bound $\Rightarrow$ sup, inf exist. Since there are sequences in $f(S)$ conv to each, closed $\Rightarrow$ elem of $f(S)$)
$\sup f(S) \in f(S)$ means there exists $x_1 \in S$ such that $f(x_1) = \sup f(S) = \max f(S)$
$x_0 \in S$ such that $f(x_0) = \inf f(S) = \min f(S)$
Thus if $x \in S$, $f(x) \in f(S)$, $f(x_0) \le f(x) \le f(x_1)$

Ex $f: [-1, 1] \to \mathbb{R}$ $f(x) = x^2$



$f([-1,1]) = [0,1]$ $\min = 0 = f(0)$ $\max = 1 = f(1)$ (or $f(-1)$)
$0 = f(0) \le f(x) \le f(1) = 1$ for all $x \in [-1,1]$



$f: (-1,1) \to \mathbb{R}$ $f((-1,1)) = [0,1)$
$\sup [0,1) = 1 \neq f(x)$ for any $x \in (-1,1)$

Thm (Intermediate Value Theorem) Let $I \subseteq \mathbb{R}$ be an interval (a,b), $[a,b]$, etc.
$f: I \to \mathbb{R}$ continuous, $a, b \in I$, $a < b$. If y is between $f(a)$ and $f(b)$ ($f(a) < y < f(b)$ or $f(b) < y < f(a)$)
Then exists $x \in (a,b)$ such that $f(x) = y$.

P.C.

Ex $f:(0,1)\to\mathbb{R}$, $f(x)=\frac{1}{x}$

$f(0,1)=(1,\infty)$

we can't extend this function to 0, it has no sup. what's the difference

Earlier: $x_0\in[-1,1]$, ... let $\varepsilon>0$, $\delta=\min\{1,\frac{\varepsilon}{2|x_0|+1}\}$, $|x_0-x|<\delta\Rightarrow|x^2-x_0^2|<\varepsilon$

Since $x_0\in[-1,1]$, $\frac{\varepsilon}{2|x_0|+1}\geq\frac{\varepsilon}{3}$. So if $\delta<\min\{1,\frac{\varepsilon}{3}\}$, $|x-x_0|<\delta\Rightarrow|x^2-x_0^2|<\varepsilon$

$f:(0,1)\to\mathbb{R}$, $f(x)=\frac{1}{x}$, is cont.

Let $x_0\in(0,1)$. Let $\varepsilon>0$. Choose $\delta=\min\left\{\frac{|x_0|}{2},\frac{\varepsilon x_0^2}{2}\right\}$

If $x\in(0,1)$, $|x-x_0|<\delta$, $(x_0-\delta<x<x_0+\delta)$

$$|f(x)-f(x_0)|=\left|\frac{1}{x}-\frac{1}{x_0}\right|=\frac{|x-x_0|}{xx_0}\leq\frac{|x-x_0|}{x_0(x_0-\delta)}<\frac{\delta}{|x_0|^2/2}\leq\varepsilon.$$

So f is cont. but δ depends on x_0, and [illegible]

can't be removed (as $x_0\to0$, $\delta\to0$; no $\delta>0$ small enough to work for all x_0)

Def $f:S\to\mathbb{R}$ is uniformly continuous if for all $\varepsilon>0$ there exists $\delta>0$ such that $x,y\in S$, $|x-y|<\delta$ implies $|f(x)-f(y)|<\varepsilon$

(a property of the whole domain S; can pick δ without knowing x_0)

So $f:[-1,1]\to\mathbb{R}$, $f(x)=x^2$ is uniformly cont.

$f:(0,1)\to\mathbb{R}$, $f(x)=\frac{1}{x}$ is not:

Choose $\varepsilon=1$. Let $\delta>0$. If $\delta\leq1$, choose $x=\sqrt{\frac{\delta}{2}}$, $y=x-\frac{\delta}{2}$.

Then $|x-y|=\frac{\delta}{2}<\delta$ but $\left|\frac{1}{x}-\frac{1}{y}\right|=\frac{|x-y|}{xy}\geq\frac{\delta/2}{x^2}=1$

(if $\delta>1$, choose $x=\frac{1}{\sqrt{2}}$, $y=\frac{1}{\sqrt{2}}-\frac{1}{2}$; then $|x-y|<1$ and $\left|\frac{1}{x}-\frac{1}{y}\right|\geq1$)

Thm. If S is s.c. and $f: S \to \mathbb{R}$ is cont, $f: S \to \mathbb{R}$ is uniformly contin.

Proof. Assume, towards a contradiction, that S is s.c., $f: S \to \mathbb{R}$ is contin, but $f: S \to \mathbb{R}$ is not uniformly contin.
Then exists $\varepsilon > 0$ such that for all $\delta > 0$, there are $x, y \in S$, $|x-y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

For each $n \in \mathbb{N}$, ~~step~~ $\delta = 1/n$, then $x_n, y_n \in S$ s.t. $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.
Since S is s.c., (x_n) has a subseq (x_{n_k}) conv to $x \in S$.
So $x_{n_k} - \frac{1}{n_k} < y_{n_k} < x_{n_k} + \frac{1}{n_k}$, $y_{n_k} \to x$ as well.
Thus by contin, $\lim_{k \to \infty} f(x_{n_k}) - f(y_{n_k}) = f(x) - f(x) = 0$.
But $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$, a contradiction.

Thm If $f: S \to \mathbb{R}$ is uniformly contin and (s_n) is a Cauchy sequence in S, $(f(s_n))$ is Cauchy.

Proof Let (s_n) be a Cauchy sequence in S. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $x, y \in S$, $|x-y| < \delta$ imply $|f(x) - f(y)| < \varepsilon$. Choose $N \in \mathbb{R}$ such that $n, m > N$ imply $|s_n - s_m| < \delta$. Then $n, m > N$ imply $|f(s_n) - f(s_m)| < \varepsilon$, so $(f(s_n))$ is Cauchy.

For ex, $\frac{1}{n} \to 0$, is Cauchy, but $\frac{1}{(1/n)} = n$ is not Cauchy.