Derivatives

<u>Def</u> Let $I \subset \mathbb{R}$ be an open interval, $a \in I$, $f: I \to \mathbb{R}$.

1) Then "difference quotient" $\dfrac{f(x)-f(a)}{x-a} : I \setminus \{a\} \to \mathbb{R}$.

2) f is <u>differentiable at a</u> if $\lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$ exists.

If the limit exists, it is called $f'(a) \in \mathbb{R}$.

3) If f is differentiable ~~at~~ at a for all $a \in I$, f is <u>differentiable</u>. Then $f': I \to \mathbb{R}$ is a function.

<u>Example</u> $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Let $a \in \mathbb{R}$

$\dfrac{f(x)-f(a)}{x-a} : \mathbb{R} \setminus \{a\} \to \mathbb{R}$

$= \dfrac{x^2-a^2}{x-a} = x+a$. $\quad \lim\limits_{x \to a} x+a = a+a = 2a$.

So $f'(a) = 2a$.

(renaming variable name, $f'(x) = 2x$.)

<u>Thm</u> Let I be an open interval, $f: I \to \mathbb{R}$ differentiable at $a \in I$. Then f is continuous at a.

Idea: If $\dfrac{f(x)-f(a)}{x-a}$ close to $f'(a)$, $|f(x)-f(a)|$ close to $|x-a| f'(a) < \delta f'(a)$

$< \varepsilon$.

<u>Proof</u> Assume f is diff. at a

Let $\varepsilon > 0$. There exists δ_1 such that if $x \in I - \{a\}$, $|x-a| < \delta_1$,

then $\left| \dfrac{f(x)-f(a)}{x-a} - f'(a) \right| < \dfrac{\varepsilon}{2}$.

Choose $\delta_2 = \min\left\{\frac{\varepsilon}{2|f'(a)|}, 1, \delta_1\right\}$.

Then if $x \in I$, $|x-a| < \delta_2 \le \delta_1$

$$\left|\frac{f(x)-f(a)}{x-a}\right| \le \frac{\varepsilon}{2} + |f'(a)|$$

$$|f(x)-f(a)| \le \frac{\varepsilon}{2}|x-a| + |f'(a)||x-a|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

let I be an open interval

Derivative Theorem: let $f, g: I \to \mathbb{R}$ be diff at a.
let $c \in \mathbb{R}$, $h: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be $h(x) = \frac{1}{x}$.

1) cf is diff at a and $(cf)' = c(f'(a))$
2) $f+g$ is diff at a and $(f+g)' = f'(a) + g'(a)$
3) fg is diff at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
4) h is diff for all $a \in \mathbb{R} \setminus \{0\}$ and $h'(a) = -\frac{1}{a^2}$.

Proof sketch

Apply limit thm to

1) $$\frac{cf(x) - cf(a)}{x-a} = c\left(\frac{f(x)-f(a)}{x-a}\right) \rightsquigarrow c\, f'(a)$$

2) $$\frac{f(x)+g(x) - f(a) - g(a)}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \rightsquigarrow f'(a)+g'(a)$$

3) $$\frac{f(x)g(x) - f(a)g(a)}{x-a} = \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a}$$

$$= \left(\frac{f(x)-f(a)}{x-a}\right)g(x) + f(a)\left(\frac{g(x)-g(a)}{x-a}\right) \rightsquigarrow f'(a)g(a) + f(a)g'(a)$$

continuous at a

4) $$\frac{\frac{1}{x} - \frac{1}{a}}{x-a} = \frac{1}{x-a}\,\frac{a-x}{xa} = -\frac{1}{xa} \rightsquigarrow -\frac{1}{a^2}.$$

Rudin 5.5

Ross 28.4

Chain Rule

Let I_1 be an open interval, $f : I_1 \to \mathbb{R}$ diff. at $a \in I_1$.
Let I_2 be an open interval, $f(I_1) \subseteq I_2$, $g : I_2 \to \mathbb{R}$ diff at $f(a)$.
Then $g \circ f : I_1 \to \mathbb{R}$ is diff at a and
$$(g\circ f)'(a) = g'(f(a))\, f'(a).$$

Idea

$$\frac{(g\circ f)(x) - (g\circ f)(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \; \frac{f(x) - f(a)}{x - a}$$

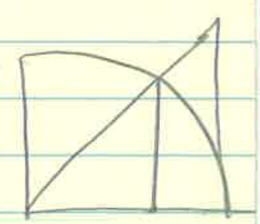
$$\downarrow \qquad\qquad\qquad\qquad \overset{?}{\downarrow}$$

$$(g\circ f)'(a) \qquad\qquad\qquad g'(f(a))\, f'(a)$$

Case I: There exists an int. $I \subset I_1$, $a \in I$, s.t. $f(x) \neq f(a)$ for $x \in I \setminus \{a\}$. Let $s_n \in I$, $s_n \to a$.
Then $f(s_n) \to f(a)$ because f is cont at a, and
$$\frac{g(f(s_n)) - g(f(a))}{f(s_n) - f(a)} \to g'(f(a)) \text{ because } \lim_{x \to f(a)} \frac{g(x) - g(f(a))}{x - f(a)} = g'(f(a))$$
thus
$$\left(\frac{g(f(s_n)) - g(f(a))}{f(s_n) - f(a)}\right)\left(\frac{f(s_n) - f(a)}{s_n - a}\right) = \frac{(g\circ f)(s_n) - g\circ f(a)}{s_n - a}$$
$$\to g'(f(a))\, f'(a).$$

Case II: If no such interval exists, then $\exists\, s_n \to a$ such that $f(s_n) = a$ for all $n \in \mathbb{N}$. Show $(g\circ f)'(a)$ and $f'(a) = 0$.

Ex: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$. Then $f'(x) = \cos(x)$.



$\sin\cos < x < \tan$

$\cos(x) < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$ $\qquad \sin < \frac{x}{\cos} < \frac{\sin}{\cos^2}$

If $x_n \to 0$, $\cos(x_n), \frac{1}{\cos(x_n)} \to 1$, $\frac{x_n}{\sin(x_n)} \to 1$, so $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.
$(\sin'(0) = 1)$

$$\lim_{x \to 0} \frac{\sin(x) - \sin(a)}{x - a} = \frac{\sin(a)\cos(x-a) + \cos(a)\sin(x-a)}{x - a} \to 0 + \cos(a).$$