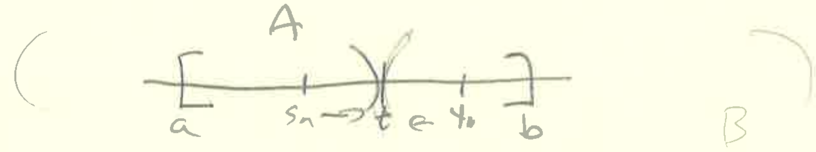

Recall We showed E not an interval $\Rightarrow$ E not connected.

Step 2 Assume E is an interval and E is disconnected (towards contradiction).
Let $A, B \subseteq \mathbb{R}$ be open sets as in def.; $a \in A \cap E$, $b \in B \cap E$.
Then $[a,b] \subseteq E$.
$[a,b] \cap A$ is a nonempty set with lower bound a and upper bound b.
Let $t = \sup([a,b] \cap A)$. Then $a \le t \le b$.

- There is a sequence (s_n) in $[a,b] \cap A$ converging to t.
- Since $s_n \in E \cap A$, $s_n \notin B$, and (s_n) is a sequence in $\mathbb{R} \setminus B$, a closed set. Since $s_n \to t$, $t \in \mathbb{R} \setminus B$.
- Then $t \neq b$, so $t < b$. Define $x_n = t + \frac{b-t}{n}$.

Then $a \le t < x_n \le b$, and $x_n \to t$.
Since $t < x_n$, $x_n \notin [a,b] \cap A$, so $x_n \in \mathbb{R} \setminus A$, a closed set.
Then $t \in \mathbb{R} \setminus A$. So $t \notin A \cup B$, but $t \in E$, a contradiction. ∎

Note: The proof is identical if we use $S = E$ as the ambient space rather than $E \subset S = \mathbb{R}$.

Note $\mathbb{R}$ is connected. Suppose $A \subseteq \mathbb{R}$ is open and closed.
Then $B = \mathbb{R} - A$ is open; $\mathbb{R} \subseteq A \cup B$, $A \cap B = \emptyset$.
So either $A \cap \mathbb{R} = A$ or $B \cap \mathbb{R} = B$ is empty
i.e. $A = \emptyset$ or $A = \mathbb{R}$.

Thm Let $f: S \to S^*$ be continuous, and $E \subseteq S$ connected.
Then $f(E)$ is connected.

Proof Assume $f(E)$ is not connected, let A, B be the open sets as in the definition. Then $f^{-1}(A)$, $f^{-1}(B)$ are open sets in S.
$E \subseteq f^{-1}(A) \cup f^{-1}(B)$ ($x \in E$, $f(x) \in A \cup B$, $f(x) \in A$ or $f(x) \in B$)
$E \cap f^{-1}(A) \cap f^{-1}(B) = \emptyset$ (if $x \in E$, $f(x) \in f(E)$, $f(x) \notin f(A) \cap f(B)$)
$E \cap f^{-1}(A) \neq \emptyset$ ($f(E) \cap A \neq \emptyset$)
$E \cap f^{-1}(B) \neq \emptyset$

Cor 1 Let $f: S \to \mathbb{R}$ be continuous, $E \subseteq S$ connected. Then $f(E)$ is an interval. In particular, if $a, b \in E$, y between $f(a)$ and $f(b)$ then there exists $x \in E$ such that $f(x) = y$.

Proof Connected sets in $\mathbb{R}$ are intervals; if $f(a) < y < f(b)$, $f(a), f(b) \in f(E)$ so $y \in f(E)$.

Cor 2 (IVT) $E \subset \mathbb{R}$ an interval, $f: E \to \mathbb{R}$ contin, $a, b \in E$, $a < b$ and y between $f(a)$ and $f(b)$, then there is a $x \in (a,b)$ such $f(x) = y$.

Proof Since $a, b \in E$, $[a,b] \subseteq E$; consider the restriction to $[a,b]$. $f: [a,b] \to \mathbb{R}$ is contin, and $[a,b]$ is connected, so $f([a,b])$ is an interval, and $y \in f([a,b])$. So $y = f(x)$ for $x \in [a,b]$.

Compactness

Let $E \subseteq S$.

An *open cover* of E is a collection C of open sets such that for every $x \in E$, $x \in U$ for some $U \in C$.
i.e. $E \subseteq \bigcup_{U \in C} U$

A *finite subcover* of C is a subset $C' \subset C$, $C' = \{U_1, \dots, U_n\}$ ~~such that~~ which is still a cover of E;
i.e. if $x \in E$, $x \in U_i$ for some $i \in \{1, \dots, n\}$
i.e. $E \subseteq U_1 \cup \dots \cup U_n$

Defn $E \subseteq S$ is compact if every open cover C of E has a finite subcover.

Example $C = \{(1,2), (\frac{1}{2}, 2), (\frac{1}{3}, 2), \dots, (\frac{1}{n}, 2), \dots\}$ is an open cover of $E = (0, 2)$.

But a finite subset of C is of the form $C' = \{(\frac{1}{n_1}, 2), \dots, (\frac{1}{n_k}, 2)\}$ for $n_1 < n_2 < \dots < n_k$. Then $\frac{1}{n_k} \in (0,2)$, but not in any of the sets in C'. So C has no finite subcover of E. E is not compact.

(Note $(0,2)$ has finite cover: ~~finite~~ on $\{(0,2)\}$,
or $\{(-\infty,1) \cup (0,\infty)\}$
or $\{\mathbb{R}\}$.
But the specific cover C has no finite subcover ?.)

C is not an open cover of $[0,2]$; $0,2$ are not elements of any $U \in C$. We have to add open sets!

$D = C \cup \{(-1,1), (1,3)\}$.
Then $\{(-1,1), (\tfrac{1}{2},2), (1,3)\} \subset D$ is a finite subcover of $[0,2]$.

We can easily prove many of the same theorems for S.C. with this def

Thm Let $f: E \to S^{+}$ be cont, $E \subseteq S$ compact. (Ross 21.4)
Then ~~Th~~
1) $f(E)$ is compact
2) f is uniformly continuous.

Ross 13.12, 13.13

Thm If $E \subset S$ is compact, it is closed and bounded (easy)
Thm If $E \subseteq \mathbb{R}^K$ is closed and bounded, it is compact (harder)

But also, it turns out (in a metric space S)
Thm E is compact if and only if E is sequentially compact
compact $\Rightarrow$ S.C. is reasonable
S.C. $\Rightarrow$ compact is harder.