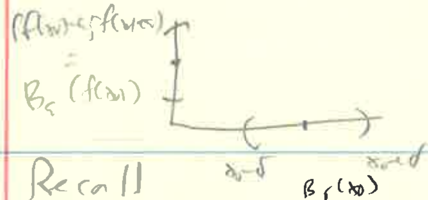


Recall

S a metric space with distance function d

S^* a metric space with distance fun d^*

$E \subseteq S$

$f: E \to S^*$ is continuous at $x_0 \in E$ if for all $\varepsilon > 0$ there exist $\delta > 0$ such that

$x \in E$, $d(x, x_0) < \delta$ implies $d^*(f(x), f(x_0)) < \varepsilon$.

$$x \in E \cap B_\delta(x_0) \text{ implies } f(x) \in B_\varepsilon(f(x_0))$$
$$f(E \cap B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$$

Def "inverse image" or "preimage"

$f: E \to S^*$, $F \subseteq S^*$,
$$f^{-1}(F) = \{x \in E \mid f(x) \in F\}$$

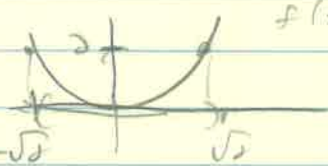
inverse func may not exist

not $f^{-1}(f(1,\sqrt{2})) = f^{-1}(1,2) = (1,\sqrt{2}) \cup (-\sqrt{2},-1)$

$f^{-1}f(E) \supseteq E$

$f(f^{-1}($

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$



$f^{-1}(1,2) = (-\sqrt{2},-1) \cup (1,\sqrt{2})$

$f^{-1}((0,\infty)) = \mathbb{R}$

$f^{-1}((-2,-1)) = \emptyset$

$f(f^{-1}((-2,-1))) = f(\emptyset) = \emptyset$

$f(f^{-1}(E)) \subseteq E$

$E = S$ whole metric space, implied

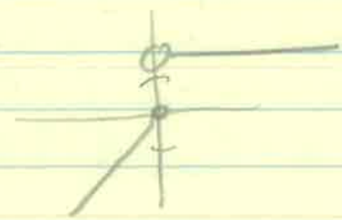
Thm $f: S \to S^*$ is continuous if and only if for every open set $U \subseteq S^*$, $f^{-1}(U)$ is an open set in S

Proof Step 1 Assume that if $U \subseteq S^*$ is open then $f^{-1}(U) \subseteq S$ is open.
Let $x_0 \in S$, $\varepsilon > 0$. $B_\varepsilon(f(x_0))^*$ is an open set, (HW), so $f^{-1}(B_\varepsilon(f(x_0)))$ is an open set. Since $d(f(x_0), f(x_0)) = 0 < \varepsilon$, $f(x_0) \in B_\varepsilon(f(x_0))$, so $x_0 \in f^{-1}(B_\varepsilon(f(x_0)))$. Thus there is $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\varepsilon f(x_0))$, so $f(B_\delta(x_0)) \subset B_\varepsilon f(x_0)$.
This proves f is continuous.

Step 2 Assume f is continuous. Let $U \subseteq S^*$ be a open set.
If $f^{-1}(U)$ is empty, it is open. If not, let $x_0 \in f^{-1}(U)$. Then

$f(x_0) \in U$, so there exists $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subset U$.
By continuity, there exists $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)) \subseteq U$
Then $B_\delta(x_0) \subset f^{-1}(U)$. So $f^{-1}(U)$ is open.

Even simpler, $f(x) = \begin{cases} 1 & x > 0 \\ x & x \leq 0 \end{cases}$



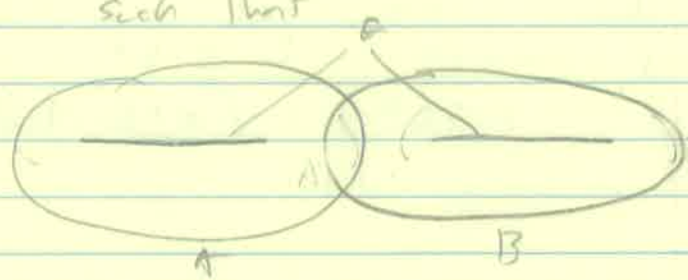
open

$f^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) = \left(-\frac{1}{2}, 0\right]$ is not open.

Connectedness

Def A set $E \subseteq S$ is disconnected if there exist open sets $A, B \subset S$ such that

1) $E \subseteq A \cup B$
2) $A \cap B \cap E = \emptyset$
3) $A \cap E \neq \emptyset$
4) $B \cap E \neq \emptyset$



E is connected if it is not disconnected; i.e., no such A, B exist.

Ex 1) $(0,1) \cup (3,4)$ is disconnected; $A = (-\infty, 2)$ $B = (2, \infty)$

2) $\mathbb{Q}$ is disconnected; $A = (-\infty, \sqrt{2})$, $B = (\sqrt{2}, \infty)$

not a sup

we can't do this with an interval

Thm $E \subseteq \mathbb{R}$ is connected if and only if E is an interval

Fact E is an interval $\Longleftrightarrow$ for all $x, y \in E$ and $c \in \mathbb{R}$ such that $x < c < y$, $c \in E$

(a,b), $[a,b)$, $[a,b]$, (a,∞) etc.

(Hw)

Proof of Thm

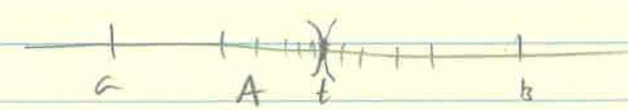
Step 1 Assume E is not an interval. Then there exist $x, y \in E$ and $c \in \mathbb{R}$ such that $x < c < y$ and $c \notin E$.
Let $A = (-\infty, c)$, $B = (c, \infty)$. Then
$E \subseteq \mathbb{R} \setminus \{c\} = A \cup B$, $A \cap B = \emptyset$, $x \in A \cap E$, $y \in B \cap E$. So E is disconnected.

Step 2 Assume E is an interval, and, towards a contradiction E is disconnected by open sets A, B satisfying 1–4. Let $a \in E \cap A$, $b \in E \cap B$.

Since E is an interval, $[a,b] \subseteq E$

Then $[a,b] \cap A$ is nonempty and bounded; let $t = \sup([a,b] \cap A)$.
Since $a \in [a,b] \cap A$, $a \le t$; since b is an upper bound for $[a,b] \cap A$, $t \le b$. Thus $t \in E$.
Then there is a seq (s_n) in $[a,b] \cap A$ converging to t. Since $s_n \in A \cap E$, $s_n \notin B$, and $s_n \in \mathbb{R} \setminus B$, a closed set. So $t \in \mathbb{R} \setminus B$. In particular, $t < b$.

Let $x_n = t + \frac{b-t}{n}$. Then $t < x_n \le b$, so $x_n \in [a,b]$ but $x_n \notin [a,b] \cap A$, so $x_n \notin A$, and $x_n \in \mathbb{R} \setminus A$, a closed set. Since $x_n \to t$, so $t \in \mathbb{R} \setminus A$. But $t \in E$, and $E \subseteq A \cup B$, a contradiction.



Thm Let $f : S \to S^*$ be continuous, $E \subseteq S$ connected. Then $f(E)$ is connected.

Proof Assume $f(E)$ is not connected; let $A, B \subseteq S^*$ satisfy 1–4.
Then $f^{-1}(A)$, $f^{-1}(B)$ are open sets, $f(E) \subseteq A \cup B$ so $E \subseteq f^{-1}(A) \cup f^{-1}(B)$.
If $x \in E \cap f^{-1}(A) \cap f^{-1}(B)$ then $f(x) \in f(E) \cap A \cap B$, so $E \cap f^{-1}(A) \cap f^{-1}(B) = \emptyset$;
Since $f(E) \cap A$ and $f(E) \cap B$ are nonempty, $f^{-1}(A) \cap E$ and $f^{-1}(B) \cap E$ are nonempty.
So E is not connected.

Cor Let $f : S \to \mathbb{R}$ be cont, $E \subseteq S$ conn. Then $f(E)$ is an interval.
~~In particular~~, If $a, b \in E$, $f(a) < y < f(b)$, then there is $x \in E$ such that $f(x) = y$.

Inter. value th

__Cor.__ Let $I \subseteq \mathbb{R}$ be an interval, $f\colon I \to \mathbb{R}$ continuous. ~~That~~ $a, b \in I$ and y is between $f(a)$ and $f(b)$, then there exists $x \in I$ such that $f(x) = y$.

__Proof__ Consider $I = S$ to be the metric space.