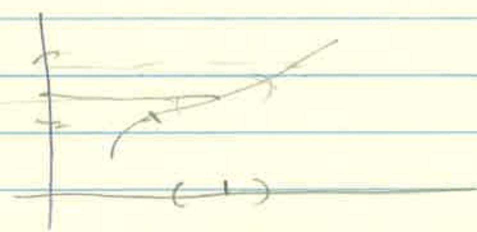
Thm ($\varepsilon$-$\delta$ property)

Given $f: S \to \mathbb{R}$, $x_0 \in \mathbb{R}$ such that a sequence in $S$ converges to $x_0$,
then $\lim_{x \to x_0} f(x) = L$ if and only if

$\varepsilon$-$\delta$ prop ( for all $\varepsilon > 0$ there exists $\delta > 0$ such that
$x \in S \setminus \{x_0\}$, and $|x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$.
(or $x \in S$, $0 < |x - x_0| < \delta$)

Corollary

Given $f: S \to \mathbb{R}$, $x_0 \in S$, $f$ is continuous at $x_0$
if and only if for all $\varepsilon > 0$ there exists $\delta > 0$
such that $x \in S$, $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.



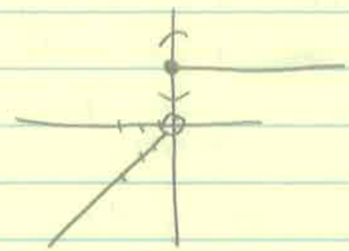
Proof of Thm

Step 1 Assume $\varepsilon$-$\delta$ property; prove limit.
Let $(s_n)$ be a sequence in $S \setminus \{x_0\}$, $s_n \to x_0$.
Let $\varepsilon > 0$. There exists $\delta > 0$ such that $x \in S \setminus \{x_0\}$, $|x - x_0| < \delta$ implies $|f(x) - L| <$
There exists $N \in \mathbb{N}$ such that $n > N$ implies $|s_n - x_0| < \delta$,
which in turn implies $|f(s_n) - L| < \varepsilon$. So $f(s_n) \to L$.

Step 2 Assume $\varepsilon$-$\delta$ property does not hold; prove $\lim \neq L$.
There exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists
$x \in S \setminus \{x_0\}$ $|x - x_0| < \delta$ but $|f(x) - L| \geq \varepsilon$.
Thus for each $n \in \mathbb{N}$, setting $\delta = 1/n$, there exists $s_n \in S \setminus \{x_0\}$
such that $|s_n - x_0| < 1/n$ but $|f(s_n) - L| \geq \varepsilon$.
Thus $x_0 - 1/n < s_n < x_0 + 1/n$, so $s_n \to x_0$. But there exists no $n \in \mathbb{N}$
such that $|f(s_n) - L| < \varepsilon$, so $f(s_n) \not\to L$.

Example $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = \begin{cases} x & \text{if } x<0 \\ 1 & \text{if } x\ge 0 \end{cases}$$



f is not continuous at $x_0 = 0$.

Def $s_n = -1/n$ converges to $0$ but since $-\frac{1}{n} < 0$

$f(-\frac{1}{n}) = -\frac{1}{n}$ conv to $0 \neq f(0) = 1$

(Note $\lim_{x\to 0} f(x)$ does not exist; $f(\frac{1}{n}) = 1 \to 1$)

ε-δ proof Choose $\varepsilon = \frac{1}{2}$, let $\delta > 0$, and choose $0 < \frac{1}{n} < \delta$

Then $|-1/n - 0| = \frac{1}{n} < \delta$ but $|f(-1/n) - f(0)| = |-\frac{1}{n} - 1| > 1/2$.

Exp $f:\mathbb{R}\to\mathbb{R}$, $f(x) = x^2$ is contin, by ε-δ

Let $\varepsilon > 0$, choose $\delta = \min\left\{1, \frac{\varepsilon}{(2|x_0|+1)}\right\}$

If $x\in\mathbb{R}$, $|x - x_0| < \delta$, then $x < x_0 + \delta \le |x_0| + |\delta|$

$$\begin{aligned} |f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0||x - x_0| &\le (|x| + |x_0|)|x - x_0| \\ &< (2|x_0| + \delta)\delta \\ &\le (2|x_0| + 1)\delta \\ &\le \varepsilon \end{aligned}$$

Fact 1) if $f: S \to \mathbb{R}$ is continuous at $x_0$, and $E \subseteq S$ such that $x_0 \in E$, then $f: E \to \mathbb{R}$ is contin at $x_0$

Def Given $f: S \to \mathbb{R}$, $E \subseteq S$, $f(E) = \{ f(x) \mid x \in E \}$
"the image of $E$"

Def Given $f: S \to \mathbb{R}$, $g: T \to \mathbb{R}$, such that $f(S) \subseteq T$, then $g \circ f : S \to \mathbb{R}$ has $g \circ f(x) = g(f(x))$.

Thm Given $f: S \to \mathbb{R}$, $g: T \to \mathbb{R}$ such that $f(S) \subseteq T$

a) $x_0 \in \mathbb{R}$ such that there's a sequence in $S$ converging to $x_0$.
If $\lim_{x \to x_0} f(x) = L \in T$ and $g$ is continuous at $L$ then $\lim_{x \to x_0} g \circ f(x) = g(L)$

b) if $f$ is cont. at $x_0$ and $g$ is cont. at $f(x_0)$, $g \circ f$ is cont. at $x_0$.

Proof Let $(s_n)$ be a sequence in $S \setminus \{x_0\}$. Then $(f(s_n))$ is a sequence in $f(S) \subseteq T$ such that $f(s_n) \to L \in T$. Since $g$ is continuous at $L$, $g \circ f(s_n) = g(f(s_n)) \to g(L)$.

Def $f: S \to \mathbb{R}$ is bounded if $f(S)$ is bounded.

$f: (0,1) \to \mathbb{R}$, $f(x) = x$ : $f((0,1)) = (0,1)$ is bounded
$f: (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ : $f((0,1)) = (1, \infty)$ not bounded.

Thm If $S$ is sequentially compact and $f: S \to \mathbb{R}$ is continuous, $f(S)$ is sequentially compact.

Proof Let $(s_n)$ be a sequence in $f(S)$. Then for each $n \in \mathbb{N}$, since $s_n \in f(S)$, we can choose $x_n$ in $S$ such that $f(x_n) = s_n$. Since $S$ is s.c., $(x_n)$ has a subseq $(t_k)$ which converges to $t \in S$. Since $f$ is continuous, $f(t_k) \to f(t) \in f(S)$. So $s_{n_k} = f(x_{n_k}) = f(t_k) \to f(t) \in S$ and $S$ is sequentially compact.