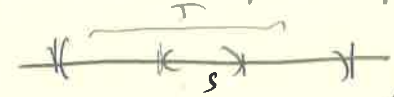
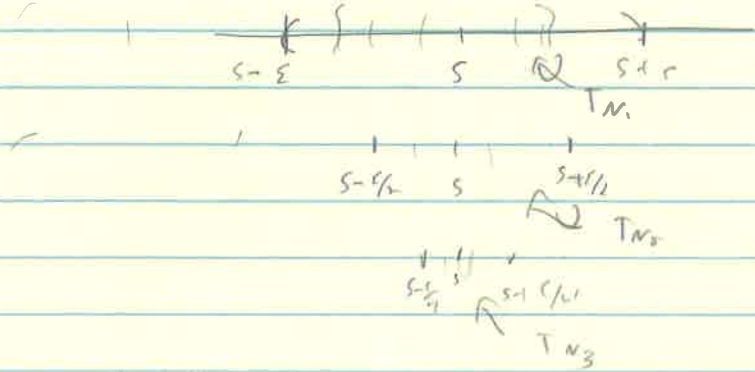
Recall: If $S \subseteq T$, $\inf T \le \inf S \le \sup S \le \sup T$



Recall: s_n increasing & bounded, $s_n \to s$ & $s_n \le s$

Recall a "tail" of (s_n) is a set $T_N = \{s_n \mid n > N\}$.
(s_n) converges if for all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $T_N \subset (s-\varepsilon, s+\varepsilon)$



$n > N_1$, $|s_n - s| < \varepsilon$

$n > N_2$, $|s_n - s| < \frac{\varepsilon}{2}$

$n > N_3$, $|s_n - s| < \frac{\varepsilon}{4}$

T_N "fits" in smaller sets as N gets larger
so $\inf T_N \to s \leftarrow \sup T_N$

Lim sup / Lim inf

Let (s_n) be a bounded sequence. Then there exist $m, M \in \mathbb{R}$ such that
$$m \le s_n \le M \quad \text{for all } n \in \mathbb{N}.$$

For all $N \in \mathbb{N}$, the tail $T_N = \{s_n \mid n > N\}$ is bounded by m and M, so
$$m \le \underbrace{\inf T_N}_{:= u_N} \le \underbrace{\sup T_N}_{:= v_N} \le M.$$

Since $T_{N+1} \subseteq T_N$, $u_N \le u_{N+1} \le v_{N+1} \le v_N$
So $(u_N)_{N=1}^{\infty}$ is an increasing sequence
$(v_N)_{N=1}^{\infty}$ is a decreasing sequence.
Both are bounded

Definition Let (s_n) be a bounded sequence.
$$\limsup s_n = \lim_{N\to\infty} v_N = \lim_{N\to\infty} \sup\{s_n \mid n > N\}$$

$$\liminf s_n = \lim_{N\to\infty} u_N = \lim_{N\to\infty} \inf\{s_n \mid n > N\}.$$

So $u_N \le v_N$, $\liminf s_n \le \limsup s_n$.
$(\lim (v_N - u_N) \ge 0)$

Recall $\boxed{u_N \le \liminf s_n \le \limsup s_n \le v_N}$

Theorem Let (s_n) be a bounded sequence.
$(s_n) \to s$ if and only if $\liminf s_n = \limsup s_n = s$.

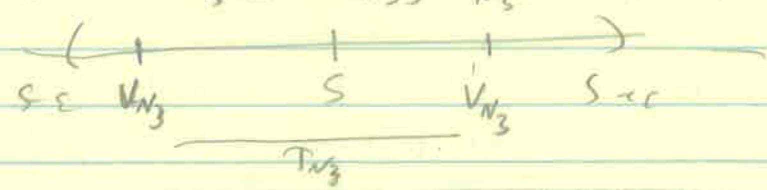
Proof **Step 1** Assume $s_n \to s$. We prove that $\limsup s_n = \liminf s_n = s$.
We know $\liminf s_n \le \limsup s_n$
Let $\varepsilon > 0$. There exists $N_0 \in \mathbb{N}$ such that $T_{N_0} = \{s_n \mid n > N_0\} \subset (s-\varepsilon, s+\varepsilon)$.
Thus $V_{N_0} = \sup T_{N_0} \le s + \varepsilon$.
So (V_N) is decreasing, $\limsup s_n = \lim_{N \to \infty} V_N \le V_{N_0} \le s + \varepsilon$.

Since this is true for all $\varepsilon > 0$, $\limsup s_n \le s$.
A similar argument shows $\liminf s_n \ge s \ge \limsup s_n$, so all 3 are equal.

Step 2 Assume $\liminf s_n = \limsup s_n = s$. We prove $s_n \to s$.
Let $\varepsilon > 0$.
As usual, let $T_N = \{s_n \mid n > N\}$, $U_N = \inf T_N$, $V_N = \sup T_N$.
Then $V_N \to \limsup s_n = s$ and
$U_N \to \liminf s_n = s$.

So there exists N_1 such that for $N > N_1$, $|V_N - s| < \varepsilon$, and $V_N < s + \varepsilon$
there exists N_2 such that for $N > N_2$, $|U_N - s| < \varepsilon$, and $U_N > s - \varepsilon$.
Let $N_3 > \max\{N_1, N_2\}$. If $n > N_3$, then
$s - \varepsilon < U_{N_3} \le s_n \le V_{N_3} < s + \varepsilon$, so $|s_n - s| < \varepsilon$. ∎

(i.e. $T_{N_3} \subseteq [U_{N_3}, V_{N_3}] \subset (s-\varepsilon, s+\varepsilon)$)



Definition A seq (s_n) is *Cauchy* if for all $\varepsilon > 0$,
there exists $N \in \mathbb{R}$ such that $n, m > N$ implies $|s_n - s_m| < \varepsilon$.

Idea If $s_n \to s$, for large n, s_n is very close to s. So
If n, m both large, s_n, s_m very close to s, and to each other.

So we expect Convergent $\Rightarrow$ Cauchy, Indeed, $\Leftrightarrow$.

<u>Lemma</u>

If (s_n) is Cauchy, it is bounded

<u>Proof</u>

Assume (s_n) is Cauchy.

Then exists $N \in \mathbb{R}$ such that for $n, m > N$, $|s_n - s_m| < 1$ $(\varepsilon = 1)$.

Choose $n_0 \in \mathbb{N}$, $n_0 > N$. For all $n \in \mathbb{N}$, if

if $n > n_0$, then $|s_n - s_{n_0}| < 1$, so $s_n < s_{n_0} + 1$

If $n \leq n_0$, then $s_n \in \{s_1, \dots, s_{n_0}\}$, so $s_n \leq \max\{s_1, \dots, s_{n_0}\}$.

Thus $\{s_n \mid n \in \mathbb{N}\}$ is bounded above by $\max\{s_{n_0} + 1, \max\{s_1, \dots, s_{n_0}\}\}$.

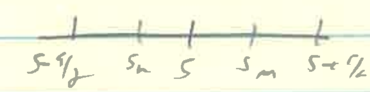
<u>Theorem</u> A seq (s_n) converges if and only if it's Cauchy!

<u>Proof</u> <u>Step 1</u> Assume $s_n \to s$. We show (s_n) is Cauchy.

Let $\varepsilon > 0$. Then exists $N \in \mathbb{R}$ such that $n > N$ implies $|s_n - s| < \varepsilon/2$.

Then if $n, m > N$, $|s_n - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.

So (s_n) is Cauchy



<u>Proof</u> Step 2 Assume (s_n) is Cauchy. We show $s_n \to s$ for some $s \in \mathbb{R}$.

First, since (s_n) is Cauchy we know s_n is bounded. So lim sup, lim inf exist,

and we can show $\limsup s_n = \liminf s_n$

Let $\varepsilon > 0$. There exists $N \in \mathbb{R}$ such that

$n, m > N$ implies $|s_n - s_m| < \varepsilon/2$.

Fix one value $m_0 > N$.

If $n > N$, then $s_n < s_{m_0} + \varepsilon/2$, so

So $s_{m_0} + \varepsilon/2$ is an upper bound for $T_N = \{s_n \mid n > N\}$.

Thus $\sup T_N = v_N \leq s_{m_0} + \varepsilon/2$.

Similarly, $\inf T_N = u_N \geq s_{m_0} - \varepsilon/2$.

Thus $s_{m_0} - \varepsilon/2 \leq u_N \leq \liminf s_n \leq \limsup s_n \leq v_N \leq s_{m_0} + \varepsilon/2$

Thus $\limsup s_n - \liminf s_n \le (s_{m_0} + \varepsilon/2) - (s_{m_0} - \varepsilon/2) = \varepsilon$.
Since this is true for all $\varepsilon > 0$, $\liminf s_n = \limsup s_n$