for HW: IK>S Cauchy ? rel res?

value $\sqrt{a}-b = \frac{\sqrt{a}-b(\sqrt{a}+b)}{\sqrt{a}+b} = \frac{a-b^2}{\sqrt{a}+b}$

Ch 18, 4 Lots of examples.

Ross 9.2 – 9.6

Limit theorems

Let $s_n \to s$ and $t_n \to t$. Let $k \in \mathbb{R}$

1) (ks_n) conv to ks
2) $(s_n + t_n)$ conv to $s+t$
3) $(s_n t_n)$ conv to st
4) If $t_n \neq 0$ for all $n \in \mathbb{N}$ $t \neq 0$ the $\left(\frac{1}{t_n}\right)$ conves to $\frac{1}{t}$

Example

$\frac{2n^2 - 7n}{n^2 - n + 8} \to 2$

$(\div n^2)$ $\frac{2 - \frac{7}{n}}{1 - \frac{1}{n} + \frac{8}{n^2}}$

$(1,1,1,\ldots)$ conv to 1

$1 - \frac{1}{n} + \frac{8}{n^2} \neq 0$ $\lim\left(1 - \frac{1}{n} + \frac{8}{n^2}\right) = \lim 1 - \lim \frac{1}{n} - 8\left(\lim \frac{1}{n}\right)\lim\left(\frac{1}{n}\right)$

$= 1 - 0 - 8(0)(0) = 1$

$\lim\left(2 - \frac{7}{n}\right) = 2 - 7\lim\frac{1}{n} = 2 - 7(0) = 2$

$\lim \frac{2 - \frac{7}{n}}{1 - \frac{1}{n} + \frac{8}{n^2}} = \lim\left(2 - \frac{7}{n}\right) \lim\left(\frac{1}{1 - \frac{1}{n} + \frac{8}{n^2}}\right) = 2 \cdot \frac{1}{1} = 2$

Proof Ideas

1) $|ks_n - ks| = |k|\,|s_n - s|$
2) $|s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t|$
3) $|s_n t_n - st| = |s_n t_n - st_n + st_n - st| \leq |s_n - s|\,|t_n| + |s|\,|t_n - t|$
4) $\left|\frac{1}{t_n} - \frac{1}{t}\right| = \frac{|t - t_n|}{|t|\,|t_n|}$ replace with lower bound

need upper bound

$(\frac{1}{n})$ is bdd, $0 < \frac{1}{n} \le 1$

Def A seq (s_n) is bdd if its set of values $\{s_n\}$ is bounded

Thm If (s_n) converges, (s_n) is bounded.

(Converse false: (s_n) may be bdd, (s_n) of values $(1, 3, 1, \ldots)$)

Proof Assume $s_n \to s$. Choose $\varepsilon = 1$; there exists $N \in \mathbb{R}$ such that if $n > N$, $|s_n - s| < \varepsilon = 1$, so $s_n < s+1$.

$m \in \mathbb{N}$, $m \le N < m+1$ (or 0 if $N \le 0$)

$m \quad N \quad m+1$

Let $M = \max\{s_1, \ldots, s_m\}$.

If $n \le m$, $s_n \in \{s_1, \ldots, s_m\}$, and $s_n \le M$

If $n > m$ $\qquad s_n < s+1$

So $s_n \le \max\{M, s+1\}$ for all $n \in \mathbb{N}$,

and $\{s_n\}$ is bdd above. Bdd below is similar.

Thm If $a, b \in \mathbb{R}$, $a \le s_n \le b$ for all $n \in \mathbb{N}$,

and $s_n \to s$, then $a \le s \le b$.

(Squeeze thm)

Thm If $a_n \le s_n \le b_n$ for all $n \in \mathbb{N}$, $a_n \to s$ and $b_n \to s$,

then $s_n \to s$

Important Limits (Ross 9.7)

1) If $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$

2) If $|a| < 1$, $\lim_{n\to\infty} a^n = 0$

3) $n^{1/n} \to 1$

4) If $a > 0$, $\lim_{n\to\infty} a^{1/n} = 1$

A sequence (s_n) is increasing if $s_n \le s_{n+1}$

A sequence (s_n) is decreasing if $s_n \ge s_{n+1}$

A sequence is monotone if it is increasing or decreasing.

<u>Thm</u> A bounded monotone sequence converges.

If (s_n) is bounded and increasing, then $s_n \le \lim s_n$

(decreasing) $s_n \ge \lim s_n$

<u>Proof</u> Let (s_n) be bounded and increasing.

Then $\{s_n \mid n \in \mathbb{N}\}$ is nonempty and bounded.

Let $r = \sup\{s_n \mid n \in \mathbb{N}\}$.

We claim that $s_n \to r$.

Let $\varepsilon > 0$. Since $r - \varepsilon < r$, there exists s_{n_0} such that $r - \varepsilon < s_{n_0} \le r$. Choose $N = n_0$. If $n > n_0$, then

$$r - \varepsilon < s_{n_0} \le s_n \le r$$

so $|s_n - r| < \varepsilon$.

$s_n \le \lim s_n = r$ by definition of r.

Def Let (s_n) be a sequence. A **tail** of (s_n) is a set of the form

$$T_N = \{ s_n \mid n > N \}$$

for some $N \in \mathbb{R}$.

$s_n \to s$ if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that

$$T_N \subset (s - \varepsilon, s + \varepsilon).$$