Let S be a metric space with distance function d

A $E \subset S$ is

<u>Closed</u>

If (s_n) is a sequence in E and $s_n \to s$ then $s \in E$

<u>open</u>

$S \setminus E$ is closed

$S \setminus E$ is open

If $s_0 \in E$, there exists $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ such that $B_\varepsilon(s_0) \subset E$

$$\{ r \in S \mid d(r, s_0) < \varepsilon \}$$

(is $S = \mathbb{R}$, $\{ r \in \mathbb{R} \mid |r - s_0| < \varepsilon \} = (s_0 - \varepsilon, s_0 + \varepsilon)$)

$E \subseteq S$ is sequentially compact if every seq (s_n) in E has a subsequence converging to an element of E.

<u>Thm</u> If E is S.C., E is closed and bounded

<u>But</u> there are closed and bounded sets which are <u>not</u> S.C.

If $S = (0,1)$, $E = S = (0,1)$, then E is closed and bounded ($S = B_{\frac{1}{2}}(\frac{1}{2})$). But $(\frac{1}{n})$ has no convergent subseq (Bolzano-Weierstrass <u>does not hold</u>).

<u>Series</u> informal

Given any list of numbers $a_1, a_2, a_3, \dots, a_k, \dots$ in $\mathbb{R}$

$\sum a_k$ represent the seq (s_n), $s_n = \sum_{k=1}^{n} a_k$ "partial sums"

$\sum_{n=1}^{\infty} a_n$

If $s_n \to s$ we say $\sum a_k = s$

(Similar $\sum_{k=m}^{\infty} a_k$ represent the seq $(s_n)_{n \geq m}$, $s_n = \sum_{k=m}^{n} a_k$)

Fact let $m_1 < m_2$. $\sum_{k=m_1}^{\infty} a_k$ conv iff $\sum_{k=m_2}^{\infty} a_k$ conv

why: let $S_n = \sum_{k=m_1}^{n} a_k$ $S_n' = \sum_{k=m_2}^{n} a_k$ then

$$S_n = S_n' + C, \text{ where } c = \sum_{k=m_1}^{m_2-1} a_k$$

Fact $\sum a_n$ conv if and only if it is Cauchy

$\iff$ for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t.

$n, m > N$ implies $|S_n - S_m| < \varepsilon$

assume $n > m$ $n, m > N$ implies $\left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$

Cor If $\sum a_k$ conv, $a_k \to 0$

Proof. let $\varepsilon > 0$. Then exists N such that for $m+1 > N$

$$\left| \sum_{k=m+1}^{m+1} a_k \right| = \left| a_{m+1} \right| < \varepsilon$$

Thus if $n > N+1$, then $(n-1) > N$ and $|a_n| < \varepsilon$

The converse is not true: $\frac{1}{n} \to 0$ but $\boxed{\sum \frac{1}{n} \text{ does not converge}}$

Limit comparison test

If $|b_k| \le a_k$ and $\sum a_k$ converges then $\sum b_k$ converges

Proof let $\varepsilon > 0$. Then exists $N \in \mathbb{R}$ such that $n > m > N$ implies $\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$

then $\left| \sum_{k=m+1}^{n} b_k \right| \le \sum_{k=m+1}^{n} |b_k| \le \sum_{k=m+1}^{n} a_k = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$.

So $\sum b_k$ is Cauchy.

Def $\sum a_k$ converges absolutely if $\sum |a_k|$ converges

Thm If $\sum a_k$ conv absolutely, $\sum a_k$ converges

Pf ...

The best way to show a sequence converges is to compare to a geometric sequence

Point Let $|a|<1$. Then $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$

Proof $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \Rightarrow \frac{1}{1-a}$ as $a^{n+1} \to 0$.

The ratio and root test helps do this

Ratio test Let $a_k \neq 0$ for all $k \in \mathbb{N}$

If $\limsup \left|\frac{a_{k+1}}{a_k}\right| < 1$, $\sum a_k$ converges absolutely

If $\liminf \left|\frac{a_{k+1}}{a_k}\right| > 1$ or does not exist, $\sum a_k$ does not converge

Root test

If $\limsup |a_k|^{1/k} < 1$, $\sum a_k$ converges absolutely

If $\limsup |a_k|^{1/k} > 1$ or does not exist, $\sum a_k$ does not converge

Proof Root test $T_N = \{|a_k|^{1/k} : k > N\}$

Since $\sup T_N \to \limsup |a_k|^{1/k} < 1$, there exists $N_0 \in \mathbb{N}$

$r := \sup T_{N_0} < 1$.

Thus for $k > N_0$, $|a_k|^{1/k} < r$, or $|a_k| < r^k$. So $\sum_{k=N_0+1}^{\infty} r^k$

Converges, $\sum_{k=N_0+1}^{\infty} |a_k|$ converges.