Def A set $E \subseteq \mathbb{R}$ is *sequentially compact* if every sequence in E has a subsequence converging to an element of E.

Thm $E \subseteq \mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

Proof (sequentially compact)

Step 1 Assume E is closed and bounded. Prove E is SC.
Let (s_n) be a sequence in E. Since E is bounded, (s_n) is bounded, so there is a subseq (t_k), $t_k \to t \in \mathbb{R}$ by Bolzano-Weierstrass. Since (t_k) is a sequence in E and E is closed, $t \in E$.

Step 2 Assume E is not closed. Prove E is not SC.
There exists a sequence (s_n) in E such that $s_n \to s$ but $s \notin E$. Since any subseq of (s_n) converges to $s \notin E$, E is not S.C.

Step 3 Assume E is not bounded above. Prove E is not SC.
For each $n \in \mathbb{N}$, n is not an upper bound for E, so there exists $s_n \in E$ such that $s_n > n$. Let (t_k) be a subseq of (s_n). Let $M \in \mathbb{R}$, and choose $K \in \mathbb{N}$, $K > M$. Then $t_K = s_{n_K} \geq n_K \geq K > M$. So M is not an upper bound for (t_k). Thus (t_k) is not bounded, and does not converge.

Many of the ideas + theorems we have discussed rely only on the notion of 'distance', and apply to the situation where 'distance' makes sense.

Def A *metric space* is a set S together with a distance fn d:
For $a, b \in S$, $d(a,b) \in \mathbb{R}$ such that

1) $d(a,b) \geq 0$ and $d(a,b) = 0$ if and only if $a = b$
2) $d(a,b) = d(b,a)$
3) $d(a,b) \leq d(a,c) + d(c,b)$ for all $c \in S$.

Δ inequality [illegible]

Example

1) $\mathbb{R}$ is a metric space, with $d(a,b) = |a-b|$

[illegible] so far

2) any subset of $\mathbb{R}$ is a metric space w/ same d
3) $\mathbb{R}^2$ is a metric space with $d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$
4) f, g bounded fns $\mathbb{R} \to \mathbb{R}$, $d(f,g) = \sup\{|f(x)-g(x)| \mid x \in \mathbb{R}\}$.

Let S be a metric space with distance function d

Def 1) A sequence is an infinite list of elements of S $(s_n) = (s_1, s_2, s_3, \ldots)$, $s_n \in S$.
2) $s_n \to s \in S$ if for all $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$), there exists $N \in \mathbb{R}$ such that $n > N$ implies $d(s, s_n) < \varepsilon$.

Thm Limits are unique

Notation For $s \in S$, $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$), $B_\varepsilon(s) = \{r \in S \mid d(r,s) < \varepsilon\}$
(if $S = \mathbb{R}$, $B_\varepsilon(s) = (s-\varepsilon, s+\varepsilon)$)
So $d(r,s) < \varepsilon \iff r \in B_\varepsilon(s)$

2) 1) A subset $E \subseteq S$ is bounded if there are $s_0 \in S$, $M > 0$ such that $E \subseteq B_M(s_0)$; (so $d(s, s_0) < M$ for all $s \in E$)
2) A sequence is bounded if its set of values is bounded

Thm 1) A convergent sequence is bounded
2) If $s_n \to s$, any subsequence of (s_n) converges to s.

Def A seq (s_n) in S is Cauchy if for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $n, m > N$ implies $d(s_n, s_m) < \varepsilon$.

Thm Convergent sequences are Cauchy (A ineq)

But Not all Cauchy seq converge
Ex Let $S = (0, 2)$, $d(a,b) = |a-b|$ (ignore rest of $\mathbb{R}$)
$(\frac{1}{n})$ is Cauchy: let $\varepsilon > 0$. Choose $N = \frac{2}{\varepsilon}$. If $n, m > N$ then
$|\frac{1}{n} - \frac{1}{m}| \leq |\frac{1}{n}| + |\frac{1}{m}| = \frac{1}{n} + \frac{1}{m} < \frac{2}{N} = \varepsilon$.
But $(\frac{1}{n})$ does not conv to any $s \in (0, 2)$.

A metric space is called complete if all Cauchy seqs converge. So $\mathbb{R}$ is a complete m.s. (so is $\mathbb{R}^k$).

But Not every bounded seq has a convergent subseq ($(\frac{1}{n})$ in $(0,2)$ again)