<u>Recall</u> Let (s_n) be a bounded sequence, $r := \limsup s_n$.
Let $\varepsilon > 0$.

c) the set $\{ n \mid |s_n - r| < \varepsilon \}$ is unbounded

<u>Theorem</u> Let (s_n) be a bounded seq, $r := \limsup s_n$. There exists a subseq (t_k) of (s_n) such that $t_k \to r$.

<u>Proof</u> We define $t_k = s_{n_k}$ one element at a time

n_1: The set $\{ n \mid |s_n - r| < 1 \}$ is nonempty; choose n_1 such that $|s_{n_1} - r| < 1$ and set $t_1 = s_{n_1}$.

n_2: The set $\{ n \mid |s_n - r| < \frac{1}{2} \}$ is unbounded; choose $n_2 > n_1$ such that $|s_{n_2} - r| < \frac{1}{2}$, and set $t_2 = s_{n_2}$

⋮

n_k: the set $\{ n \mid |s_n - r| < \frac{1}{k} \}$ is unbounded; choose $n_k > n_{k-1}$ such that $|s_{n_k} - r| < \frac{1}{k}$ and set $t_k = s_{n_k}$.

So $n_1 < n_2 < n_3 < \dots n_{k-1} < n_k < \dots$, (t_k) is a subsequence of (s_n).

So $|t_k - r| = |s_{n_k} - r| < \frac{1}{k}$, $s - \frac{1}{k} < t_k < s + \frac{1}{k}$

and $t_k \to s$ by the squeeze theorem. ∎

Thus every bounded sequence has a convergent subseq (Bolzano-Weierstrass)

~~Another approach~~

<u>Thm</u> Every sequence contains a monotonic subsequence Ross 11.4

Since bounded monotone sequences converge, this also implies Bolzano-Weierstrass

<u>Thm</u> If (s_n) is a bounded sequence, and (t_k) is a subsequence such that $t_k \to t$, then $\liminf s_n \le t \le \limsup s_n$.

<u>Proof</u> Ross 11.8

Open, Closed, and Compact Sets

Let $S \subseteq \mathbb{R}$

Lingo We say a seq (s_n) is "in" S if $s_n \in S$ for all $n \in \mathbb{N}$.

Two equivalent definitions

A set $E \subseteq \mathbb{R}$ is

	Closed	Open
def 1	If (s_n) is a seq in E and $s_n \to s$, then $s \in E$	If $\mathbb{R} \setminus S$ is closed
def 2	If $\mathbb{R} \setminus S$ is open	If for all $s \in E$ there exist $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \subseteq E$

Note: A set can be neither open nor closed, or both.

Example $\{s \in \mathbb{R} : a \le s \le b\} = [0,1]$ then is closed (seq, (s_n))
If (s_n) is a seq in $[0,1]$ then $0 \le s_n \le 1$ for all $n \in \mathbb{N}$,
so $0 \le s \le 1$, or $s \in [a,b]$

Example $(0,1)$ is open (ε, $[0,1]$)

1) Let $s \in (0,1)$, and let $\varepsilon = \min\{s, 1-s\} > 0$.
If $r \in (s-\varepsilon, s+\varepsilon)$, then $r < s+\varepsilon \le s+1-s = 1$
$r > s-\varepsilon \ge s-s = 0$
So $r \in (0,1)$.

$\mathbb{R}$ is closed (if $s_n \to s$, $s \in \mathbb{R}$ by definition) and open (as $\varepsilon > 0$
$(s-\varepsilon, s+\varepsilon) \subset \mathbb{R}$ by definition)

We check the claim

$\mathbb{R} \setminus [a,b] = (-\infty, a) \cup (b, \infty)$ is open (HW)

$\mathbb{R} \setminus (0,1) = (-\infty, 0] \cup [1, \infty)$ is closed (HW)

So HW.

$[a,b]$ is not open: $(b-\varepsilon, b+\varepsilon)$ contains $b + \frac{\varepsilon}{2} > b$ for any $\varepsilon > 0$ (also, can't be not closed)

$(0,1)$ is not closed: $\frac{1}{n} \in (0,1)$, $\frac{1}{n} \to 0 \notin (0,1)$.

$[0,1)$ is neither open nor closed (HW)

Thm The definitions are equivalent

i.e. S is open (def 2) iff $\mathbb{R} \setminus S$ is closed (def 1).

Proof [illegible]).

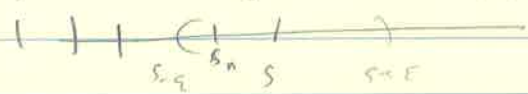
Step 1 If E is not closed, $\mathbb{R} \setminus E$ is not open.

(Thus if $\mathbb{R} \setminus E$ is open, E is closed)

Then exists (s_n) in E such that $s_n \to s$ and $s \notin E$. So $s \in \mathbb{R} \setminus E$.

Let $\varepsilon > 0$. Then exists s_n such that $|s_n - s| < \varepsilon$. So

$s_n \in (s-\varepsilon, s+\varepsilon)$, so $s_n \in E$, $(s-\varepsilon, s+\varepsilon) \not\subseteq \mathbb{R} \setminus E$



Thus $\mathbb{R} \setminus E$ is not open.

Step 2 If E is not open, $\mathbb{R} \setminus E$ is not closed.

Since E is not open, there exists $s \in E$ such that for all $\varepsilon > 0$, $(s-\varepsilon, s+\varepsilon) \not\subseteq E$. For each $n \in \mathbb{N}$, set $\varepsilon = 1/n$, and choose $s_n \in (s - 1/n, s + 1/n)$ such that $s_n \in \mathbb{R} \setminus E$.

Then $s - \frac{1}{n} < s_n < s + \frac{1}{n}$, so $s_n \to s \in E$. So $\mathbb{R} \setminus E$ is not closed.

Def A set $E \subset \mathbb{R}$ is sequentially compact if every sequence in E has a subsequence converging to an element of E.

Thm (Heine-Borel) E is sequentially compact if and only if it is closed and bounded.

Proof

St1 Closed and bound => sequentially compact

Let E be closed and bounded. Let (s_n) be a sequence in E. Since (s_n) is bounded, it has a convergent subsequence (t_k), $t_k \to t$ (Bolzano-Weierstrass). Since (t_k) is a sequence in E, and E is closed, $t \in E$.

St2 Not (closed and bounded) => Not sequentially comp

Assume E is not closed. Then there exists a seq (s_n) in E such that $s_n \to s$ and $s \notin E$. Since any subseq of (s_n) also converges to s, E is not sequentially compact.

Now Assume E is not bounded above. For each $n \in \mathbb{N}$, n is not an upper bound of E, so we can choose $s_n \in E$ such that $s_n \geq n$. If (t_k) is a subsequence of (s_n), $t_k \geq s_{n_k} \geq n_k \geq k$, so (t_k) is not bounded, and does not converge. So E is not sequentially compact.