Subsequences

Def A sequence $(t_k)_{k=1}^{\infty}$ is a subsequence of $(s_n)_{n=1}^{\infty}$ if for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $t_k = s_{n_k}$ and $n_1 < n_2 < n_3 < \dots$

i.e. we take an strict incr list of natural numbers, and pick out certain elem of (s_n)

$(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8)$

$n_1 = 2$ $\quad n_2 = 4$ $\quad n_3 = 5$ $\quad n_4 = 8$

$t_1 = s_2$ $\quad t_2 = s_4$ $\quad t_3 = s_5$ $\quad t_4 = s_8$

$(t_k) = (s_2, s_4, s_5, s_8)$

example $\quad n = 1 \; 2 \; 3 \; 4 \; 5 \; 6$

$(s_n) = (1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \dots)$

$(t_k) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ $\quad n_k = 2k$

$(t_k) = (1, 1, 1, \dots)$ $\quad n_k = 2k-1$

Note: $(t_k) = (2, 6, 4, 8, 10, 12)$ is not a subseq of $(s_n) = (1, 2, 3, 4, 5, 6, 7, 8, \dots)$

wrong order

$t_2 = s_6 \quad t_3 = s_4$

$n_2 = 6 > n_3 = 4$

$(t_k) = (2, 2, 4, 6, 8, \dots)$ is not a subseq of $(s_n) = (1, 2, 3, 4, 5, 6, \dots)$

repeat $\quad$ once

$t_1 = s_2 \quad t_2 = s_2$

$n_1 = 2 = n_2 = 2$

Lemma $n_k \geq k$.

Take $k=1$: $n_1 \in \mathbb{N}$ so $n_1 \geq 1$. Now assume $n_k \geq k$, & $n_{k+1} > n_k \geq k$,

$n_{k+1} \geq k+1$.

Theorem If $s_n \to s$, any subseq of (s_n) converges to s.

Proof Assume $s_n \to s$, and let (t_k) be a subseq of (s_n), so $t_k = s_{n_k}$ for all $k \in \mathbb{N}$.

no ex. done in class →

Let $\varepsilon > 0$. Then exists $N \in \mathbb{R}$ such that $n > N$ imp $|s_n - s| < \varepsilon$.
If $k > N$, $n_k \geq k > N$, so $|t_k - s| = |s_{n_k} - s| < \varepsilon$.
Thus $t_k \to s$.

$\boxed{\text{Theorem (Bolzano-Weierstrass) A bounded sequ has a convergent subsequence.}}$

Let (s_n) be a bdd seq and

Lemma Let $r_0 = \limsup s_n$

1) Let $r < r_0$. The set $\{n \mid r < s_n\}$ is not bounded
2) Let $r > r_0$. The set $\{n \mid r \leq s_n\}$ is finite
3) Let $\varepsilon > 0$ be. $\{n \mid |s_n - r_0| < \varepsilon\}$ is not bdd

Proof

1) Let $r < r_0$ and assume $\{n \mid r < s_n\}$ is bdd above by N. If $n > N$, then $s_n \leq r$, so r is an upper bd of $T_N = \{s_n \mid n > N\}$. Thus $V_N \leq r < r_0$, a contradiction.

2) Let $r > r_0$. Since the (V_N) conv to r_0, there exists $N_0 \in \mathbb{N}$ such that $|V_{N_0} - r_0| < r - r_0$ and $V_{N_0} < r$. Then $n > N_0$ imp $s_n \leq V_{N_0} < r$, so $\{n \mid s_n \geq r\} \subset \{1, 2, \ldots, N_0\}$, a finite set.

3) Let $\varepsilon > 0$. Assume $\{n \in \mathbb{N} \mid |s_n - r_0| < \varepsilon\} = \{n \in \mathbb{N} \mid r_0 - \varepsilon < s_n < r_0 + \varepsilon\}$ is bdd above. By 2), $\{n \in \mathbb{N} \mid r_0 + \varepsilon \leq s_n\}$ is finite, and thus bounded above. Thus
$\{n \in \mathbb{N} \mid r_0 - \varepsilon < s_n\} = \{n \in \mathbb{N} \mid r_0 - \varepsilon < s_n < r_0 + \varepsilon\} \cup \{n \in \mathbb{N} \mid r_0 + \varepsilon \leq s_n\}$
is bdd above, contra 1).

Examples

lim sup / lim inf cases

$(s_n) = (\frac{1}{n})_{n=1}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

for $N \in \mathbb{N}$

$T_N = \{ s_n \mid n > N \} = \{ \frac{1}{N+1}, \frac{1}{N+2}, \frac{1}{N+3}, \ldots \}$

$v_N = \sup T_N = \frac{1}{N+1}$

$u_N = \inf T_N = 0$

$(v_N)_{N=1}^{\infty} = (\frac{1}{N+1})_{N=1}^{\infty}$ conv to $0 = \limsup \frac{1}{n}$

$(u_N)_{N=1}^{\infty} = (0)_{N=1}^{\infty} = (0, 0, 0, \ldots)$ conv to 0

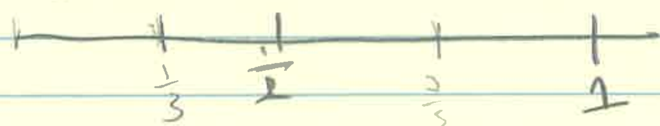
$(1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, 1, \frac{1}{6}, \ldots$

	v_N	u_N
$T_0 = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$	1	0
$T_1 = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$	1	0
$T_2 = \{1, \frac{1}{3}, \frac{1}{4}\}$	1	0
$T_3 = \{1, \frac{1}{3}, \frac{1}{4}\}$	1	0
$T_4 = \{1, \frac{1}{4}, \ldots\}$	1	0
	1	

$\limsup s_n = 1$ $\qquad$ $\liminf = 0$

does not conv.

$(s_n) = (1, \frac{1}{2}, 1, \frac{1}{3}, \ldots)$



$\{ n \mid \frac{2}{3} < s_n \} = \{1, 3, 5, \ldots\}$ infinite $\left(\frac{2}{3} < 1 = \limsup\right)$

$(s_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

$\{ n \mid \frac{2}{3} < s_n \} = \{1\}$ bound