Fun facts if $c>0$,
1) $|a-b|<c \iff b-c<a<b+c \iff a\in(b-c,b+c)$
2) If $b_1\geq b_1$ in $\ldots$ $b_1\geq a$. Then $b\geq a$
3) If $r<\sup S$, then there is $s\in S$ such that $r<s\leq\sup S$

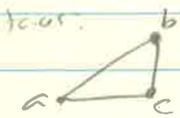
(s_n) a sequence, $s\in\mathbb{R}$

$s_n\to s$ if
For all $\varepsilon>0$ there exists $N\in\mathbb{R}$ such that $n>N$ implies $|s_n-s|<\varepsilon$

$s_n\not\to s$ if
there exist $\varepsilon>0$ such that for all $N\in\mathbb{R}$ there exists $n>N$ such that $|s_n-s|\geq\varepsilon$

<u>Fact</u> (another version of the triangle inequality)

distances:



$|a-b|\leq|a-c|+|c-b|$
(why: $|a-b|=|(a-c)+(c-b)|\leq\ldots$)

<u>Example</u> $s_n=(-1)^n$ does not converge to any $s\in\mathbb{R}$.
idea:

$-1 \qquad s \qquad 1$

For any a, either $|-1-s|$ or $|1-s|$ ≥ 1.
$s_n=1$ for even $n>N$, -1 for odd $n>N$
Let $s\in\mathbb{R}$ choose choose

<u>Proof</u> Choose $\boxed{\varepsilon=1}$. Let $N\in\mathbb{R}$.
Note that $2=|1-(-1)|\leq|1-s|+|s-(-1)|$ (choose)
so either $|1-s|\geq 1$, in which case choose n even, $n>N$, so $s_n=1$
or $|(-1)-s|\geq 1$ then choose n odd, $n>N$ so $s_n=-1$.

Then in any case $|s_n-s|\geq 1$.

<u>Example</u> $\dfrac{2n^2-7n}{n^2-n+8}\longrightarrow 2$

<u>Idea</u> Make n big enough that $\left|\dfrac{2n^2-7n}{n^2-n+8}-2\right|<\varepsilon$.

$$\left|\frac{2n^2-7n-2n^2+2n-16}{n^2-n+8}\right|=\left|\frac{-5n-16}{n^2-n+8}\right|=\left|\frac{5n+16}{n^2-n+8}\right|\leq\left|\frac{5n+16}{n^2-n}\right|$$

Q get an upper bound which is simple, like $\frac{c}{n}$, but still small if n is large

$16 \le 16n \quad (n \ge 1)$

$$\left|\frac{5n+16}{n^2-n}\right| \le \frac{21n}{n^2-n} \le \frac{21n}{n^2/2} = \frac{42n}{n^2} = \frac{42}{n} < \varepsilon$$

For large n

$n^2-n \approx n^2$, so should be $\ge \frac{n^2}{2}$

$n^2-n \ge \frac{n^2}{2}$ iff $\frac{n^2}{2} \ge n$ iff $\boxed{n \ge 2}$.

Now we can "solve" for n: If $n > \frac{42}{\varepsilon}$, and $n \ge 2$, then

$$|s_n - 2| \le \frac{42}{n} < \varepsilon.$$

Proof Let $\varepsilon > 0$. Choose $N = \max\{2, \frac{42}{\varepsilon}\}$. Then, if $n > N$

Note $n \ge 2$, so $\frac{n^2}{2} \ge n$ and $n^2-n \ge n^2/2$

$$|s_n - 2| = \left|\frac{5n+16}{n^2-n}\right| \le \left|\frac{21n}{n^2-n}\right| \le \frac{21n}{n^2/2} = \frac{42}{n} < \frac{42}{N} \le \varepsilon$$

So $\forall n > N$ we $|s_n - 2| < \varepsilon$.

Thm (Limits are unique) If $s_n \to s$ and $s_n \to t$ then $s = t$.

Proof Let $\varepsilon > 0$. Take an N_1 such that $n > N_1$ implies $|s_n - s| < \varepsilon/2$

and N_2 such that $n > N_2$ implies $|s_n - s| < \varepsilon/2$

Then for $n > \max\{N_1, N_2\}$

$$|s-t| \le |s-s_n| + |s_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $|s-t| < \varepsilon$ for all $\varepsilon > 0$, and $|s-t| \ge 0$, so $s = t$.

HW1, #4

(countable)

Thm Let $S \subset \mathbb{R}$, $r = \sup S$. Then there is a (s_n) such that $s_n \in S$ for all n and $s_n \to r$.

Proof (pick the seq) For each $n \in \mathbb{N}$, $r - 1/n < r = \sup S$,

so there is an element of S, which we call s_n, such that $r - \frac{1}{n} < s_n \le \sup S$.

To show the (s_n) we show $s_n \to r$. Let $\varepsilon > 0$, choose $N = \frac{1}{\varepsilon}$. Then if $n > N$

so $r - \frac{1}{n} < s_n \le r \,(\le r + \frac{1}{n})$, so $|s_n - r| < \frac{1}{n} < \frac{1}{N} = \varepsilon$.