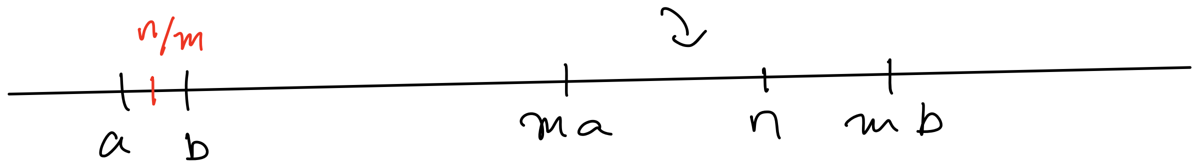
Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$)

If $a, b \in \mathbb{R}$, $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Idea multiply by large $m \in \mathbb{N}$ (Arch. Princ.) to get distance > 1



Show that there exists an integer n in between

Lemma If $r \in \mathbb{R}$, there exists $n_0 \in \mathbb{Z}$ such that $r - 1 < n_0 \leq r$

(i.e. $\lfloor r \rfloor$, integer part (r) exists).

Proof of Lemma

By the Arch. Princ. there exists $m \in \mathbb{N}$ such that $m > \max\{|r|, |r-1|\}$

Then $-m < r-1 < r < m$

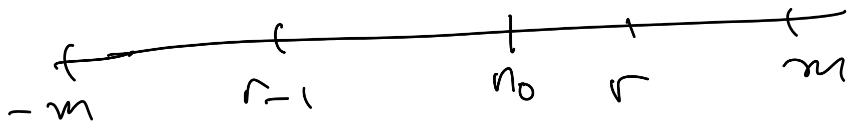
Let $S = \{n \in \mathbb{Z} \mid r-1 < n \leq m\}$

Then $S \subseteq \{-m, -m+1, \dots, m-1, m\}$

So S is finite, and nonempty $(m \in S)$

So $n_0 = \min S$. Thus $r - 1 < n_0$

and $n_0 - 1 \notin S$ $n_0 - 1 \leq r - 1$, so $n_0 \leq r$.



Proof of Density of $\mathbb{Q}$

Let $a, b \in \mathbb{R}$, $a < b$. Since $b - a > 0$, then exists $m \in \mathbb{N}$ such that $\frac{1}{b-a} < m$.

Then $1 + ma < mb$.

Use the lemma to find $n_0 \in \mathbb{Z}$ such that

$$ma < n_0 \leq 1 + ma < mb$$

So $a < \frac{n_0}{m} < b$.

Corollary (irrationals are dense in $\mathbb{R}$)

If $a, b \in \mathbb{R}$, $a < b$, then there exists $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$

Complement $\{r \in \mathbb{R} \mid r \notin \mathbb{Q}\}$

Proof

Let $a, b \in \mathbb{R}$, $a < b$. Then $a - \sqrt{2} < b - \sqrt{2}$

So there exists $s \in \mathbb{Q}$ such that $a - \sqrt{2} < s < b + \sqrt{2}$.

Then $a < s + \sqrt{2} < b$.

If $r = s + \sqrt{2} \in \mathbb{Q}$, then $\sqrt{2} = r - s \in \mathbb{Q}$, which is false. So $r \notin \mathbb{Q}$.

Sequences (ch 7, 8)

A sequences is an infinite list of numbers

$(s_m, s_{m+1}, s_{m+2}, \ldots) = (s_n)_{n=m}^{\infty}$

$m \in \mathbb{Z}$, the starting point, usually 1

$s_n \in \mathbb{R}$

element

index $\in \mathbb{Z}$

If we write (s_n), that means $(s_1, s_2, s_3, \ldots)$

start at 1

Given a sequence $(s_n)_{n=m}^{\infty}$, the <u>set of values</u> is $\{s_m, s_{m+1}, s_{m+2}, \ldots\}$

$(1, 2, 3, 4, \ldots) = (n)_{n=1}^{\infty} = (n)$

Set of values : $\mathbb{N}$

$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}) = (\frac{1}{n})_{n=1}^{\infty} = (\frac{1}{n})$

Set of values $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

order / repeats matter in sequences

$(1, 2, 1, 2, 1, 2, \ldots)$

$s_1 = 1 \quad s_2 = 2 \quad s_3 = 1, \ldots$

$(2, 1, 2, 1, 2, \ldots)$

$(1, 1, 2, 2, 2, 2, \ldots)$

set of values $\{1, 2\}$

Def A sequence (s_n) converges to $s \in \mathbb{R}$ if

$(\varepsilon \in \mathbb{R})$

For all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $n > N$ implies $|s_n - s| < \varepsilon$.

we write $s_n \to s$ or $\lim s_n = s$ or $\lim_{n\to\infty} s_n = s$
"s is the limit of (s_n)"

Idea: • Fix a "distance" $\varepsilon > 0$ (scale, error)
for large indexes $(n > N)$, s_n is close to s $(|s_n - s| < \varepsilon)$.

• N will likely depend on ε.
(smaller ε probably requires bigger N)

• prove convergence, ε is unknown, you pick N (maybe with a formula or procedure).

Example Let $s_n = 1/n$ for all $n \in \mathbb{N}$.
Then (s_n) converges to 0.
(i.e. $\frac{1}{n} \to 0$)

Proof

Let $\varepsilon > 0$. (always start this way).
Choose $N = 1/\varepsilon$. (the main work is to find this N).
If $n > N$, then $|s_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N} = \varepsilon$
So $n > N$ implies $|s_n - 0| < \varepsilon$. ∎

Process the order of proof is opposite the problem solving.

Goal: $|s_n - s| < \varepsilon$

in this case, $|s_n - s| = |\frac{1}{n} - 0| = \frac{1}{n}$

$\frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$, so if $N = \frac{1}{\varepsilon}$, then

$n > N \implies \frac{1}{n} < \varepsilon$.

Proving things don't converge

want to show (s_n) does not converge to s.

Negate:

For all $\varepsilon > 0$ (there exists N such that (n > N implies $|s_n - s| < \varepsilon$.))

Negate: For all $\varepsilon > 0$ ( ) is true

↳ There exists $\varepsilon > 0$, ( ) is false

Negate ( ) There exists N such that ( )

↳ For all N, ( ) is false.

( ) is false if there is a case where

$n > N$ but $|s_n - s| < \varepsilon$ is false.

There exists $n > N$ and $|s_n - s| \geq \varepsilon$.

There exists $\varepsilon > 0$ such that for all $N \in \mathbb{R}$, there exists $n > N$ such that $|s_n - s| \geq \varepsilon$.