<u>Ross</u> Good for examples, problems
pretty close to what we cover

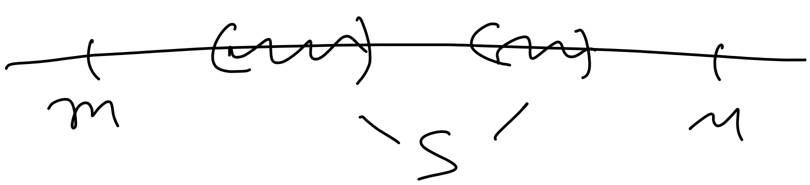
<u>Rudin</u> elegant proofs
got everything

<u>Completeness</u>

Both $\mathbb{Q}$ and $\mathbb{R}$ satisfy the ordered field axioms.
whats the difference?
$\mathbb{Q}$ has "gaps" like $\sqrt{2}$ $\mathbb{R}$ "has no gaps"
make this rigorous with upper/lower bounds

<u>Def 1</u> Let $S \subseteq \mathbb{R}$

$M \in \mathbb{R}$ is an <u>upper bound</u> for S if
$$M \geq r \quad \text{for all } r \in S$$
$m \in \mathbb{R}$ is a <u>lower bound</u> for S if
$$m \leq r \quad \text{for all } r \in S$$



<u>Examples</u> upper bounds lower bounds max min

$\{1, 16, 9\}$	anything ≥ 16	anything ≤ 1	16	1
$\{0, 1\}$	≥ 1	≤ 0	0	1
$(0, 1)$	≥ 1	≤ 0	X	X
$[0, \infty)$	X	anything ≤ 0	X	0
$\mathbb{Z}$	X	X	X	X
$\{-\frac{1}{n} \mid n \in \mathbb{N}\}$	≥ 0	≤ -1	X	-1

$= \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$

__Note__ M, m need not be in S

S can 0 or ∞ many

__Terminology__

S is __bounded above__ if it has an upper bound

S is __bounded below__ if it has a lower bound

S is __bounded__ if it has both

__Notation__ (Intervals) $a, b \in \mathbb{R}$ $\quad a \leq b$

$[a, b] = \{r \in \mathbb{R} \mid a \leq r \leq b\}$

(a, b) $\quad a < r < b$

$[a, b)$ $\quad a \leq r < b$

$(a, b]$ $\quad a < r \leq b$

$[a, \infty)$ $\quad a \leq r$

(a, ∞) $\quad a < r$

$(-\infty, a]$ $r \le a$

$(-\infty, a)$ $r < a$

$(-\infty, \infty) = \mathbb{R}$

Def 2 $S \subseteq \mathbb{R}$

1) r_0 is the maximum of S if $r_0 \in S$ and r_0 is an upper bound for S

"r naught"
"r zero"

2) r_0 is the minimum of S if $r_0 \in S$ and r_0 is a lower bound

notation: $r_0 = \max S$ $r_0 = \min S$

Note: A set has 0 or 1 max, 0 or 1 min

Note: A nonempty finite set always has a max and a min

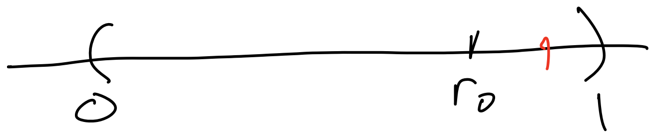
Theorem If r_0, s_0 are maxima of S, then $r_0 = s_0$

Proof Assume r_0, s_0 are maxima of S.
Thus $r_0, s_0 \in S$ and are upper bounds for S.
So $r_0 \ge s_0$ and $s_0 \ge r_0$, and $r_0 = s_0$ ∎

$(0,1)$ has no max or min

Thm If $r_0 \in (0,1)$ then r_0 is not an upper bound for $(0,1)$

Proof Assume $r_0 \in (0,1)$.



Since $r_0 < 1$, $r_0 < \frac{r_0+1}{2} < 1$

check this

$\frac{r_0}{2} < \frac{1}{2}$ $\quad \frac{r_0}{2} + \frac{r_0}{2} < \frac{r_0}{2} + \frac{1}{2}$ $\quad r_0 < \frac{r_0+1}{2}$ etc.

Therefore $\frac{r_0+1}{2} \in (0,1)$, and $> r_0$ so r_0 is not an upper bound for $(0,1)$.

Def 3 Let $S \subseteq \mathbb{R}$

$r_0 \in \mathbb{R}$ is the supremum of S if it is the least upper bound.

i.e. $r_0 = \min\{M \in \mathbb{R} \mid M \text{ is an upper bound for } S\}$

equivalently, r_0 is an upper bound and if M is an upper bound, $r_0 \le M$.

equivalently, r_0 is an upper bound, and $r < r_0$, then r is not an upper bound.

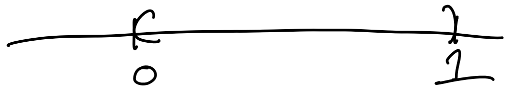
$r_0 \in \mathbb{R}$ is the infimum of S if it is the greatest lower bound

i.e.
- $r_0 = \max\{m \in \mathbb{R} \mid m \text{ is a lower bound for } S\}$

or
- r_0 is a lower bound and if m is a lower bound $r_0 \ge m$

or
- r_0 is a lower bound and if $r > r_0$ r is not a lower bound.

$(0,1)$



supremum $= 1$

infimum $= 0$

Notation: $\sup S$ (supremum) $\quad \inf S$ (infimum)

Note: a set has 0 or 1 sup / inf

Completeness Axiom: Let $S \subseteq \mathbb{R}$. If S is bounded above, then S has a supremum.

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