Homework 7

Due Tuesday, March 29 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that $h : \mathbb{R}$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x < 0 \\ g(x) & \text{if } x \geq 0 \end{cases}$$

   is continuous at 0 if and only if $f(0) = g(0)$.

2. Ross 18.5 a.

3. a) Prove that $f : [1, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous.
   b) Prove that $f : [0, 2] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous (Hint: Use the result of HW 6, problem 2).
   c) Use parts a and b to prove that $f : [0, \infty)$, $f(x) = \sqrt{x}$ in uniformly continuous.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.

5. Let $I \subset \mathbb{R}$ be a bounded nonempty set with the property that if $a, b \in I$ and $a < c < b$ then $c \in I$. Let $x = \inf I$ and $y = \sup I$.
   a) Prove that $(x, y) \subseteq I \subseteq [x, y]$.
   b) If I is closed prove that $I = [x, y]$.
   c) If I is open prove that $I = (x, y)$.

6. Let S be a metric space with distance function d. Let $x \in S$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$. Prove that $B_\epsilon(x) = \{y \in S | d(y, x) < \epsilon\}$ is an open set.

7. Let S be a metric space with distance function d. Let E and F be connected subsets of S such that $E \cap F$ is nonempty. Prove that $E \cup F$ is connected.

8. Prove that the "unit circle" $E = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ is connected. Hint: recall that sin and cos are continuous and parameterize.