Homework 5

Due Tuesday, March 1 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. a) Give a definition of an open subset of $\mathbb{R}$.
   b) Give a definition of a closed subset of $\mathbb{R}$ (make sure a and b do not give circular definitions).
   c) Given an example of a subset of $\mathbb{R}$ which is open and not closed.
   d) Given an example of a subset of $\mathbb{R}$ which is closed and not open.
   e) Given an example of a subset of $\mathbb{R}$ which is neither closed nor open.
   f) Give and example of a subset of $\mathbb{R}$ which is both open and and closed.

2. Let S be a metric space with distance function d. Let (s_n) be a sequence in S which converges to $s \in S$. Prove that (s_n) is bounded.

3. Let S be a metric space with distance function d. Let E be a sequentially compact subset of S and let F be a closed subset of S such that $F \subseteq E$. Prove that F is sequentially compact.

4. Consider the series $\sum \frac{k^2}{3^k}$.
   a) Use the ratio test to show the series converges.
   b) Use the root test to show the series converges. (Recall that $\lim n^{1/n} = 1$)
   c) Use the limit comparison test to show the series converges (Hint: Compare to a geometric series. You might need to consider only large k.)

5. Ross 14.2

6. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. See Ross 14.14; the method there can be rephrased as:

$$\text{Compare } \sum_{n=2^{k-1}}^{2^k-1} \frac{1}{n} \text{ to } \sum_{n=2^{k-1}}^{2^k-1} \frac{1}{2^k}$$

7. Ross 15.1 and 15.2.

8. a) Prove that $\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$ is a metric space with the alternate distance function

$$d'((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

   (Hint: for the triangle inequality, first proove that

$$d'((x_1, y_1), (x_2, y_2)) \leq d'((x_1, y_1), (0, 0)) + d'((x_2, y_2), (0, 0)).$$

   Then prove

$$d'((x_1, y_1), (x_2, y_2)) = d'((x_1 - a, y_1 - b), (x_2 - a, y_2 - b))$$

   for any $a, b \in \mathbb{R}$.)
   (Continued on page 2)

b) Let $\epsilon > 0$. Draw a picture of the set $B_\epsilon((0,0)) = \{(x,y) \in \mathbb{R}^2 | d'((x,y),(0,0)) < \epsilon\}$, and the set $\{(x,y) \in \mathbb{R}^2 | d((x,y),(0,0)) < \epsilon\}$, where d is the usual distance function

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

c) **If you are interested, no need to hand in:** Prove the triangle inequality for the usual distance function d on $\mathbb{R}^2$ as given above.