Homework 4

Due Tuesday, February 22 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. a) Give the definition of $\limsup$ and $\liminf$
   b) Give an example of a sequence (s_n) such that the set of values $\{s_n | n \in \mathbb{N}\}$ is infinite, $\limsup s_n = 2$, and $\liminf s_n = 1$.
   c) Given an example of a sequence (s_n) such that $\sup\{s_n | n \in \mathbb{N}\} > \limsup s_n$

2. For each sequence, find a monotone subsequence converging to the $\liminf$ or show that none exists.
   a) $((-1)^n)$
   b) $\left(\frac{1}{n}\right)$
   c) (n^2)
   d) $\left((-1)^n + \frac{1}{n}\right)$
   e) $n \cos\left(\frac{n\pi}{4}\right)$

3. Prove that
   a) $(1, \infty)$ is open and not closed.
   b) $[1, \infty)$ is closed and not open.
   c) $[0, 1)$ is neither open nor closed
   d) $\{0\} \cup \{1, 1/2, 1/3, ..., 1/n, ...\}$ is closed and not open.
   e) $\mathbb{Q}$ is neither open nor closed.

4. Let $E \subset \mathbb{R}$ be an open set. Let (s_n) be a sequence which converges to $s \in E$. Prove that there exists $N \in \mathbb{R}$ such that $n > N$ implies $s_n \in E$.

5. Let $E \subset \mathbb{R}$ be nonempty and sequentially compact. Prove that $\sup E$, $\inf E$ exist, and that $\sup E \in E$ and $\inf E \in E$.

6. For each $n \in \mathbb{N}$, let U_n be an open set and let V_n be a closed set.
   a) Prove that the union of all the U_n is open.
   b) Prove the the intersection of all the V_n is closed.
   c) Prove that for any $n \in \mathbb{N}$, $U_1 \cap U_2 ... \cap U_n$ is open.
   d) Give an example of collection of open sets U_n such that the intersection of all the U_n is not open.
   e) Give an example of collection of closed sets V_n such that the union of all the V_n is not closed.

   (problem 7 on page 2)

7. Let $a_n \in \mathbb{R}$, $a_n > 0$ for all $n \in \mathbb{N}$. Consider the sequence $\left(\frac{a_{n+1}}{a_n}\right)$. Let $C \in \mathbb{R}$ such that

$$\limsup \frac{a_{n+1}}{a_n} < C.$$

Prove that there exists $N \in \mathbb{N}$ such that for $n > N$,

$$a_n < C^{n-N} a_N.$$