Homework 3

Due Tuesday, February 15 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. a) State what it means for a sequence (s_n) to converge to $s \in \mathbb{R}$.
   b) State what it means for a sequence (s_n) to not converge to $s \in \mathbb{R}$.
   c) Give an example of a sequence (s_n) such that $s_n < 0$ for all $n \in \mathbb{N}$ and $s_n \to 0$.
   d) Give an example of a bounded sequence which does not converge.

2. Ross 8.2. This time, use limit theorems (Theorems from lecture 6; that is, Ross Theorems 9.2-9.6, and the"squeeze" theorem from in Ross Problem 8.5.)

3. a) Ross 9.1
   b) Ross 9.2

4. Prove, directly from the definition of a Cauchy sequence, that the following sequences are not Cauchy:
   a) $s_n = n$
   b) $t_n = (-1)^n$

5. For each sequence (s_n) below, find $\limsup s_n$ and $\liminf s_n$. Justify your answer.
   a) $s_n = (-1)^n$
   b) $s_n = (-1/2)^n$
   c) $s_n = (-1)^n + 1/n$

6. a) Ross, 9.12, Part a.
   b) Let $|a| \leq 1, p \in \mathbb{N}$. Prove that $\frac{a^n}{n^p} \to 0$.
   c) Let $b > 0$. Prove that $\frac{b^n}{n!} \to 0$.

7. a) Let $a < 1$. Prove that $1 + a + ... + a^n = \frac{1-a^{n+1}}{1-a}$.
   b) Let (s_n) be a sequence in $\mathbb{R}$ such that $|s_{n+1} - s_n| < 1/2^n$ for all n. Prove that s_n is Cauchy.

8. Let s_n be defined inductively by $s_1 = 2$ and $s_{n+1} = \frac{s_n}{2} + \frac{1}{s_n}$.
   a) Show that $s_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$.
   b) Prove that s_n is monotone. Prove that s_n converges.
   c) Find $\lim s_n$ and justify your answer. Hint: Consider each side of the equation above as a sequence, and find its limit using limit theorems.