Homework 11

Due Tuesday, April 26 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Ross 34.2

2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ for $x < 0$, $f(x) = x$ for $x \in [0, 1]$ and $f(x) = 4$ for $x > 1$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \int_0^x f$.
   a) Determine $F(x)$ for each $x \in \mathbb{R}$.
   b) At which $x \in \mathbb{R}$ is F continuous?
   c) At which $x \in \mathbb{R}$ is F differentiable? For those x, what is $F'(x)$?

3. Ross 34.5

4. Ross 34.7. Indicate precisely how you use change of variables, and check that all conditions of the theorem are met.

5. Let $f_n, f : S \to \mathbb{R}$ be functions, for all $n \in \mathbb{N}$. Prove that $f_n \to f$ uniformly if and only if
$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \mid x \in S\} = 0.$$

6. For $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = \left(x - \frac{1}{n}\right)^2$. Find $f : [0, 1] \to \mathbb{R}$ such that $f_n \to f$ uniformly, and prove your assertion.

7. For $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = nx^n(1 - x)$. Let $f : [0, 1] \to \mathbb{R}$ be $f(x) = 0$. Prove that $f_n \to f$ pointwise, but not uniformly. Recall $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.