Math 104 004 **Midterm** **October 14, 2021**

Name: ______________________________

SID: ______________________________

Instructions :

1. You have 80 minutes, 11:10am-12:30pm.
2. No books, notes, or other outside materials are allowed.
3. There are 5 questions on the exam. Each question is worth 10 points.
4. You need to show all of your work and justify all statements. If you need more space, use the pages at the back of the exam or come get more paper at the front of the class. If you do so, please indicate which page your solution continues on.
5. Before you begin, take a quick look at all the questions on the exam, and start with the one you feel the most comfortable solving. It is more important to do the problems well that you know how to do, than it is to finish the whole exam.
6. While attempting any problem, do write something even if you are unable to solve it completely. You may get partial credit.

(Do not fill these in; they are for grading purposes only.)

1	
2	
3	
4	
5	
Total	

1. a) (5 pts) Prove that $(0,1] \subset \mathbb{R}$ is neither open nor closed.

$(\frac{1}{n})$ is a sequence in $(0,1]$ but it converges to 0, which is not in $(0,1]$. So $(0,1]$ is not closed.

$(1+\frac{1}{n})$ is a sequence in $\mathbb{R}\backslash(0,1]$, but $1+\frac{1}{n} \to 1 \in (0,1]$. So $\mathbb{R}\backslash(0,1]$ is not closed, and thus $(0,1]$ is not open.

OR

For any $r > 0$, $1+\frac{r}{2} \in \{x \in \mathbb{R} | |x-1| < r\}$ and $1+\frac{r}{2} \notin (0,1]$, so $\{x \in \mathbb{R} | |x-1| < r\} \nsubseteq (0,1]$.

For any $r > 0$, $\min\{\frac{1}{2}, \frac{r}{2}\} \in \{x \in \mathbb{R} | |x| < r\}$ and $\min\{\frac{1}{2}, \frac{r}{2}\} \in (0,1]$. So $0 \in \mathbb{R}\backslash(0,1]$ but $\{x \in \mathbb{R} | |x| < r\} \nsubseteq \mathbb{R}\backslash(0,1]$ so $\mathbb{R}\backslash(0,1]$ is not open. Thus $(0,1]$ is not closed

b) (5 pts) Let $r \in \mathbb{R}$. Use the denseness of $\mathbb{Q}$ to prove that there exists a sequence (s_n) such that $s_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $s_n \to r$.

For each $n \in \mathbb{N}$ $r - \frac{1}{n} < r$. By the density of $\mathbb{Q}$ there is a rational number, which we call $s_n \in \mathbb{Q}$, such that $r - \frac{1}{n} < s_n < r$.

Let $\epsilon > 0$. Choose $N = \frac{1}{\epsilon}$. If $n > N$, then

$$|r - s_n| = r - s_n < r - \left(r - \frac{1}{n}\right) = \frac{1}{n} < \frac{1}{N} = \epsilon.$$

Thus $s_n \to r$.

2. (10 pts) For all $n \in \mathbb{N}$, define $s_n = \frac{1}{\sqrt{n}}$ if n is odd and $s_n = \frac{n}{n^2-1}$ if n is even.

   Find the limit of the sequence (s_n). Prove that (s_n) converges to your answer.

We prove that $s_n \to 0$.

Let $\epsilon > 0$. Choose $N = \max\{\frac{1}{\epsilon^2}, \frac{2}{\epsilon}\}$. Let $n > N$.

If n is odd,

$$|s_n - 0| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} \leq \epsilon.$$

If n is even, since $n > 1$, $n^2 > 2$ and $\frac{n^2}{2} > 1$, so

$$|s_n - 0| = \frac{n}{n^2 - 1} \leq \frac{n}{\frac{n^2}{2}} = \frac{2}{n} < \frac{2}{N} \leq \epsilon.$$

In either case $|s_n - 0| < \epsilon$, which proves that $s_n \to 0$.

3. For all $n \in \mathbb{N}$, define $t_n = 2$ if n is odd and $t_n = 2^{-n}$ if n is even.

   a) (8 pts) Find $\limsup t_n$ and $\liminf t_n$. Prove your answer.

   b) (2 pts) Does (t_n) converge? Prove your answer.

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a) The sequence is bounded by 0 and 2, so lim sup and lim inf exists.
   Let N be even. Then $\{t_n | n > N\} = \{2, 2^{-N-2}, 2, 2^{-N-4}, ...\}$.

   Then $\sup\{t_n | n > N\} = 2$ since $2^{-n} < 2$ for all $n \in \mathbb{N}$.

   We show that $\inf\{t_n | n > N\} = 0$. All elements of the set are positive, so 0 is a lower bound. Let $r > 0$. Then there exists $M \in \mathbb{R}$ such that for $m > M$, $2^{-m} < r$. There exists an even number m such that $m > \max M, N$. Then $2^{-m} \in \{t_n | n > N\}$ and r is not a lower bound for $\{t_n | n > N\}$. It follows that 0 is the greatest lower bound.

   The same claims hold when N is odd. Thus

$$\limsup t_n = \lim_{N\to\infty} \sup\{t_n | n > N\} = \lim_{N\to\infty} 2 = 2$$

$$\liminf t_n = \lim_{N\to\infty} \inf\{t_n | n > N\} = \lim_{N\to\infty} 0 = 0.$$

b) If (t_n) converges,then $\limsup t_n = \liminf t_n = \lim t_n$. Therefore, (t_n) does not converge.

4. (10 pts) Define $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x$ if $x \leq 2$ and $g(x) = x^2$ if $x > 2$. Prove that g is not continuous.

- For all $n \in \mathbb{N}$ let $x_n = 2 + \frac{1}{n}$. Then
$$\lim x_n = \lim \left(2 + \frac{1}{n}\right) = 2 + \lim \frac{1}{n} = 2.$$
Since $x_n > 2$,
$$\lim f(x_n) = \lim \left(4 + \frac{4}{n} + \frac{1}{n^2}\right) = 4 + 4 \lim \frac{1}{n} + \left(\lim \frac{1}{n}\right)\left(\lim \frac{1}{n}\right) = 4.$$
But $f(2) = 2 \neq \lim f(x_n)$. So f is not continuous at 2.

- Or, Chose $\epsilon = 2$. Let $\delta > 0$. Let $x = 2 + \frac{\delta}{2}$. Then $|x - 2| < \delta$ but since $x > 2$
$$|f(x) - f(2)| = \left|4 + 2\delta + \frac{\delta^2}{4} - 2\right| > 2.$$
So f is not continuous at 2.

5. Define $f : [1, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x^2}$.

a) (4 pts) Prove that f is uniformly continuous.

b) (4 pts) Find a sequence (x_n) in $(0, \infty)$ such that (x_n) is Cauchy but $\left(\frac{1}{x_n^2}\right)$ is not Cauchy. Prove your answer.

c) (2 pts) Prove that $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x^2}$ is not uniformly continuous.

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a) Let $\epsilon > 0$. Chose $\delta = \frac{\epsilon}{2}$. Let $x, y \in [1, \infty)$ such that $|x - y| < \delta$. Then

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \frac{|y^2 - x^2|}{x^2y^2} = \frac{|x-y|(x+y)}{x^2y^2} = |x-y|\left(\frac{1}{xy^2} + \frac{1}{x^2y}\right) < 2\delta = \epsilon.$$

b) Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ so x_n is Cauchy. But $\frac{1}{x_n^2} = n^2$ is not bounded, so it is not Cauchy.

c) If x_n is Cauchy and f is uniformly continuous, then $f(x_n)$ is Cauchy. Thus $x_n = \frac{1}{n}$ proves that f is not uniformly continuous.