Math 104 004 **Midterm** **March 10, 2022**

Name: ______________________________

SID: ______________________________

Instructions :

1. You have 80 minutes, 3:40pm-5:00pm.
2. No books, notes, or other outside materials are allowed.
3. There are 6 questions on the exam.
4. You need to show all of your work and justify all statements. If you need more space, use the pages at the back of the exam or come get more paper at the front of the class. If you do so, please indicate which page your solution continues on.
5. Before you begin, take a quick look at all the questions on the exam, and start with the one you feel the most comfortable solving. It is more important to do the problems well that you know how to do, than it is to finish the whole exam.
6. While attempting any problem, do write something even if you are unable to solve it completely. You may get partial credit.

(Do not fill these in; they are for grading purposes only.)

1	
2	
3	
4	
5	
6	
Total	

1. Give an example of each of the following objects. No justification is necessary. (2 points each)

   a) A sequence (t_k) which is a subsequence of $(s_n) = \big(2 + (-1)^n + \frac{1}{n}\big)$, such that (t_k) converges to $\limsup s_n$.

$$t_K = S_{2K} = 2 + 1 + \frac{1}{2K} \to 3$$

   b) A nonempty subset of $\mathbb{R}$ which is sequentially compact.

any closed and bounded set:

$[1,2]$, $\{3, 6, 8\}$ etc.

   c) A sequence (s_n) such that $s_n < 2$ for all $n \in \mathbb{N}$ and $s_n \to 2$.

$s_n = 2 - \frac{1}{n}$ or $s_n = \frac{2n}{n+1}$, etc.

   d) A subset of $\mathbb{R}$ which is closed and not bounded.

$\mathbb{R}$, $[1, \infty)$, $(-\infty, 0]$, $\mathbb{N}$, $\mathbb{Z}$, etc.

   e) A sequence (s_n) such that $s_n > 0$ for all $n \in \mathbb{N}$, $\limsup s_n = 4$ and $\liminf s_n = 0$.

$(s_n) = (4, 1, 4, \frac{1}{2}, 4, \frac{1}{3}, \ldots)$

or $s_n = \begin{cases} 4 & n \text{ odd} \\ \frac{1}{n} & n \text{ even} \end{cases}$

or $s_n = 2 + (-1)^n 2 + \frac{1}{n}$

2. (10 points) Find the limit s of the sequence $\left(\frac{2n^3+n}{2n^3-1}\right)$. Prove that the sequence converges to s using only the definition of convergence (no theorems or other limits).

$s = 1$.

Let $\varepsilon > 0$. Let $N = \sqrt{\frac{2}{\varepsilon}}$

If $n \in \mathbb{N}$, $n > N$ then

$$|s_n - 1| = \left|\frac{2n^3+n}{2n^3-1} - 1\right| = \left|\frac{2n^3+n-(2n^3-1)}{2n^3-1}\right| = \left|\frac{n+1}{2n^3-1}\right|$$

$$\leq \left|\frac{n+n}{2n^3-n^3}\right| = \frac{2n}{n^3} = \frac{2}{n^2} < \frac{2}{N^2} = \varepsilon.$$

since $n \geq 1$

3. (10 points) Prove that $\sqrt{2}$ is the supremum of the set

$$S = \{x \in \mathbb{Q} | x^2 < 2\}.$$

If $x > \sqrt{2}$ (>0) then $x^2 > x\sqrt{2} > \sqrt{2}^2 = 2$.

Thus if $x \in S$, $x \leq \sqrt{2}$, so $\sqrt{2}$ is an upper bound for S.

Let $s < \sqrt{2}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $s \leq \max\{s, 0\} < r < \sqrt{2}$.

Then since $r > 0$, $r^2 < r\sqrt{2} < \sqrt{2}^2 = 2$, so $r \in S$.

Thus s is not an upper bound for S, and $\sqrt{2} = \sup S$.

4. For each $n \in \mathbb{N}$ define

$$s_n = \begin{cases} 2 & \text{if } \; n \text{ is odd} \\ 4 - \frac{1}{n} & \text{if } \; n \text{ is even} \end{cases}.$$

a) (5 points) Prove that the sequence (s_n) does not converge.

b) (5 points) Prove that the series $\sum_{n=1}^{\infty} \left(\frac{1}{s_n}\right)^n$ does converge.

a) s_n has subsequences

$s_{2k} = 4 - \frac{1}{2k} \longrightarrow 4$

and $s_{2k+1} = 2 \longrightarrow 2 \neq 4$

All subsequences of a convergent sequence converge to the same limit. So s_n does not converge.

b) Since $4 - \frac{1}{n} \geq 4 - 1 = 3 > 2$, $s_n \geq 2$ for all n

and $\left|\left(\frac{1}{s_n}\right)^n\right| = \left(\frac{1}{s_n}\right)^n \leq \left(\frac{1}{2}\right)^n$.

Since $\frac{1}{2} < 1$, $\sum \left(\frac{1}{2}\right)^n$ is a convergent geometric series, so $\sum \left(\frac{1}{s_n}\right)^n$ converges by the limit comparison test.

5. (5 points) Prove that $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 + 3x & \text{if} \quad x > 2 \\ \sqrt{x} & \text{if} \quad x \leq 2 \end{cases}$$

is not continuous.

Let $s_n = 2 + \frac{1}{n}$. Then $s_n \to 2$. Since $s_n > 2$, $f(s_n) = (s_n)^2 + 3s_n = 4 + \frac{4}{n} + \frac{1}{n^2} + 6 + \frac{3}{n}$

$= 10 + \frac{7}{n} + \frac{1}{n^2}$

$\lim f(s_n) = 10 + 7\lim \frac{1}{n} + \left(\lim \frac{1}{n}\right)^2 = 10 \neq f(2) = \sqrt{2}$.

So f is not continuas at 2.

6. (5 points) Prove that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ is uniformly continuous. You can use the following facts from trigonometry without proof:

   (a) $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$

   (b) $\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $|x-y| < \delta$ then

$|f(x) - f(y)| = |\sin(x) - \sin(y)| = 2\left|\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right|$

$\leq 2\left|\sin\left(\frac{x-y}{2}\right)\right| \leq 2\left|\frac{x-y}{2}\right| = |x-y| < \delta = \varepsilon.$

since $|\cos| \leq 1$