1) Let $\varepsilon > 0$. Choose $\delta = \varepsilon$.

If $x, y \in S$ and $d(x,y) < \delta$ then

$$|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)|$$
$$\leq d(x,y) < \delta = \varepsilon.$$

2) Construct a sequence $S_n = \frac{f(n)}{n^3}$. Then $\lim S_n$

$$= \lim \frac{a_0}{n^3} + \lim \frac{a_1}{n^2} + \lim \frac{a_2}{n} + \lim a_3$$

$$= 0 + 0 + 0 + a_3 = a_3$$

So $\frac{f(n)}{n^3} \to a_3$, and for some n_1,

$\frac{f(n_1)}{n_1} > 0$, so $f(n_1) > 0$.

Similarly, $\lim \frac{f(-n)}{(-n)^3} = a_3$

So for some n_2 $\frac{f(-n_2)}{(-n_2)^3} > 0$

and $f(n_2) < 0$. By the intermediate value theorem there exists $x \in [-n_2, n_1]$ such that $f(x) = 0$.

3) Define $h : [a, b] \to \mathbb{R}$, $h(x) = f(x) - g(x)$.

Then h is continuous, $h(a) = f(a) - g(a) \geq 0$

and $h(b) = f(b) - g(b) \leq 0$.

Thus 0 is between $h(a)$ and $h(b)$,

and by the IVT there exists

$x \in [a, b]$ such that

$$0 = h(x) = f(x) - g(x).$$

c) Suppose $x, y \in E$, $x < y$.

Then there exists $r \in \mathbb{R} \setminus \mathbb{Q}$, $x < r < y$. Let $A = (-\infty, r)$ and $B = (r, \infty)$. Then A, B open

$E \subseteq \mathbb{Q} \subseteq A \cup B = \mathbb{R} \setminus \{r\}$

$A \cap B = \emptyset$

$x \in A \cap E$, $y \in B \cap E$. Thus E is not connected. So E cannot contain 2 different elements, and must have only one element.

5) a)

$$\lim_{x\to a} \frac{x^3-a^3}{x-a} = \lim_{x\to a} \frac{(x-a)(x^2+xa+a^2)}{x-a}$$

$$= \lim_{x\to a} x^2+xa+a^2 = a^2+a^2+a^2 = 3a^2$$

so $f'(2) = 12.$

b) $$\lim_{x\to a} \frac{x+2-(a+2)}{x-a} = \lim_{x\to a} 1 = 1$$ so $g'(a)=1.$

c) $$\lim_{x\to 0} \frac{x^2\cos(x)-0}{x-0} = \lim_{x\to 0} x\cos(x) = 0$$

so $f'(0) = 0.$

d) $$\lim_{x\to 1} \frac{\frac{3x+4}{2x-1} - 7}{x-1} = \lim_{x\to 1} \frac{-11x+11}{(2x-1)(x-1)}$$

$$= \lim_{x\to 1} \frac{-11}{2x-1} = -11$$ so $r'(1) = -11.$

6)

a)

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}\left(\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right)$$

$$= \lim_{x\to a} \frac{x-a}{x-a}\left(\frac{1}{\sqrt{x}+\sqrt{a}}\right) = \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}.$$

b)

$$\lim_{x\to a} \frac{x^{\frac{1}{3}}-a^{\frac{1}{3}}}{x-a} = \lim_{x\to a} \frac{x^{\frac{1}{3}}-a^{\frac{1}{3}}}{x-a} \frac{\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}\right)}{\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}\right)}$$

$$= \lim_{x\to a} \left(\frac{x-a}{x-a}\right)\frac{1}{\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}\right)} = \frac{1}{3a^{2/3}}$$

c)

$$\lim_{x\to 0} \frac{x^{1/3}}{x} = \lim_{x\to 0} \frac{1}{x^{2/3}}$$ does not exist (∞)

So f is not diff. at 0.

7) a) x, $\sin(x)$, $\frac{1}{x}$ are all diff on $(0,\infty)$ or $(-\infty,0)$

so by the product + chain rules f is diff. on $(0,\infty)$ and $(-\infty,0)$, and for $a\neq 0$

$$f'(a) = 2a\sin\left(\frac{1}{a}\right) + a^2\cos\left(\frac{1}{a}\right)\left(-\frac{1}{a^2}\right)$$
$$= 2a\sin\left(\frac{1}{a}\right) - \cos\left(\frac{1}{a}\right).$$

b) $$\lim_{x\to 0} \frac{x^2\sin\left(\frac{1}{x}\right) - 0}{x-0} = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

Since $-x \leq x\sin\left(\frac{1}{x}\right) \leq x$ and $\lim_{x\to 0} x = \lim_{x\to 0} -x = 0$,

$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ by the squeeze theorem

(if $s_n \to 0$, $s_n\sin\left(\frac{1}{s_n}\right)\to 0$.) So $f'(0)=0$.

c) Define $s_n = \frac{1}{n\pi}$. Then $s_n \to 0$

but $f'(s_n) = \frac{2}{n\pi}\sin(n\pi) - \cos(n\pi) = -\cos(n\pi)$
$$= -(-1)^n \not\to 0 = f'(0).$$

So f' is not continuous.

8)

a) Use the product rule and induction to show that $(x^n)' = nx^{n-1}$ for all $n \in \mathbb{N}$.

b) Use the fact that $\left(\frac{1}{x}\right)' = \left(-\frac{1}{x^2}\right)$ and the chain and product rules to prove the quotient rule: If $I \subseteq \mathbb{R}$ is an open interval, $f, g : I \to \mathbb{R}$ are differentiable at $a \in I$, and $g(x) \neq 0$ for $x \in I$, then

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

a) when $n = 1$ $(x)' = 1$ by the definition:

$$\lim_{x \to a} \frac{x-a}{x-a} = 1.$$

Now assume $(x^n)' = nx^{n-1}$. Then by the product rule

$$(x^{n+1})' = (x^n)'x + x^n(x)' = (nx^{n-1})x + x^n = (n+1)x^n.$$

b) $\frac{f}{g}$ is the product of f and the composition of $\left(\frac{1}{x}\right) \circ g$. Since g is differentiable and nonzero and $\frac{1}{x}$ is differentiable at $x \neq 0$, $\left(\frac{1}{x}\right) \circ g$ is differentiable by the chain rule and $\frac{f}{g}$ is differentiable by the product rule. We compute:

$$\begin{aligned}
\left(\frac{f}{g}\right)' &= \left[(f)\left(\left(\frac{1}{x}\right) \circ g\right)\right]' \\
&= f'\left(\left(\frac{1}{x}\right) \circ g\right) + f\left(\left(\frac{1}{x}\right) \circ g\right)' \\
&= \frac{f'}{g} + f\left(\left(-\frac{1}{x^2}\right) \circ g\right) g' \\
&= \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}
\end{aligned}$$