Homework 8

Due Monday, Novermber 7 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Let S be a metric space with distance function d. Fix a point $x_0 \in S$. Define $f : S \to \mathbb{R}$ by $f(x) = d(x, x_0)$. Prove that f is uniformly continuous.

2. Let $a_0, a_1, a_2, a_3 \in \mathbb{R}$ with $a_3 > 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Prove that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$.

3. Ross 18.5 a.

4. Let E be a nonempty connected subset of $\mathbb{R}$ such that $E \subset \mathbb{Q}$. Prove that E has exactly one element.

5. Ross 28.2

6. Ross 28.3

7. Ross 28.4

8. a) Use the product rule and induction to show that $(x^n)' = nx^{n-1}$ for all $n \in \mathbb{N}$.

   b) Use the fact that $\left(\frac{1}{x}\right)' = \left(-\frac{1}{x^2}\right)$ and the chain and product rules to prove the quotient rule: If $I \subseteq \mathbb{R}$ is an open interval, $f, g : I \to \mathbb{R}$ are differentiable at $a \in I$, and $g(x) \neq 0$ for $x \in I$, then
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$