1) a) Let $\varepsilon > 0$. Choose $\delta = \varepsilon$.

If $x, y \in [1, \infty)$, $|x-y| < \delta$, then since $\sqrt{x}+\sqrt{y} \geq 2$

$$|\sqrt{x}-\sqrt{y}| = \frac{|\sqrt{x}-\sqrt{y}||\sqrt{x}+\sqrt{y}|}{\sqrt{x}+\sqrt{y}} = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} < \frac{\delta}{2} < \varepsilon.$$

b) Since fcn: $\sqrt{x}$ is continuous and $[0,2]$ is sequentially compact, f is uniformly continuous.

Let $\varepsilon > 0$.

c) Since $f: [0,2] \to \mathbb{R}$ is unif. cont., there exists $\delta_1 > 0$ such that if $x, y \in [0,2]$, $|x-y| < \delta_1$, then $|\sqrt{x}-\sqrt{y}| < \varepsilon$.

Chose $\delta_2 = \min\{\varepsilon, \delta_1, 1\}$. Let $x, y \in [0, \infty)$, $|x-y| < \delta_2$, $x < y$.

If $x < 1$, then

$$y < x + \delta_2 \leq x + 1 < 2$$

so $x, y \in [0,2]$ and $|x-y| < \delta_2 \leq \delta_1$ implies $|\sqrt{x}-\sqrt{y}| < \varepsilon$.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.

Let $\epsilon > 0$. A continuous function on a closed interval is uniformly continuous, so if we change the domain to the subset $[0, 2] \subset \mathbb{R}$, we get a uniformly continuous function. Since $f : [0, 2] \to \mathbb{R}$ is uniformly continuous, there exists $\delta_1 > 0$ such that if $x, y \in [0, 2]$ and $|x - y| < \delta_1$ then $|f(x) - f(y)| < \epsilon$.

Let $\delta_2 = \min\{\delta_1, 1\}$. Now let $a, b \in \mathbb{R}$ with $|a-b| < \delta_2$. We can assume $a \leq b$. Let n be the largest integer which such that $n \leq a$. Then $a < n+1$. Since $b < a + \delta_2 \leq a + 1 < n + 2$, $a, b \in [n, n+2]$. Then $a - n, b - n \in [0, 2]$ and $|(a - n) - (b - n)| = |a - b| < \delta_2 \leq \delta_1$ so $|f(a - n) - f(b - n)| < \epsilon$. Applying the property n times, $|f(a) - f(b)| < \epsilon$.

$(f(a - n) = f(a - n + 1) = f(a - n + 2) = ... = f(a - n + n))$

3) Let $y \in B_\varepsilon(x)$. Choose $\delta = \varepsilon - d(y,x) > 0$. If $s \in B_\delta(y)$,

$$d(s,x) \le d(s,y) + d(y,x) < \delta + d(y,x) = \varepsilon.$$

So $s \in B_\varepsilon(x)$. Thus $B_\delta(y) \subseteq B_\varepsilon(x)$ proving $B_\varepsilon(x)$ is open.

Let (s_n) be a sequence in $\overline{B_\varepsilon(x)}$ such that $s_n \to t$. Let $\delta > 0$. There exists $n \in \mathbb{N}$ such that $d(s_n, t) < \delta$, so

$$d(x,t) \le d(x, s_n) + d(s_n, t)$$
$$\le \varepsilon + \delta$$

since $s_n \in \overline{B_\varepsilon(x)}$.

Since this is true for all $\delta > 0$, $d(x,t) \le \varepsilon$, and $t \in \overline{B_\varepsilon(x)}$.

4)

a) • $d'((x_1, y_1), (x_2, y_2)) \geq |x_1 - x_2| \geq 0$

If $d'((x_1, y_1), (x_2, y_2)) = 0$, then $|x_1 - x_2| = 0$

So $x_1 = x_2$, and similarly $|y_1 - y_2| = 0$ so $y_1 = y_2$. Clearly $d'((x, y), (x, y)) = 0$.

So $d'((x_1, y_1), (x_2, y_2)) = 0 \iff (x_1, y_1) = (x_2, y_2)$.

• Since $|x_1 - x_2| = |x_2 - x_1|$ and $|y_1 - y_2| = |y_2 - y_1|$, $d'((x_1, y_1), (x_2, y_2)) = d'((x_2, y_2), (x_1, y_1))$.

• triangle inequality:

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| \leq |x_1| + |x_2| \\ \text{or } |y_1 - y_2| \leq |y_1| + |y_2| \end{cases}$$

$$\leq \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\} = d((x_1, y_1), (0,0)) + d((x_2, y_2), (0,0)).$$

Then

$$d((x_1, y_1), (x_2, y_2)) = d((x_1 - x_3, y_1 - y_3), (x_2 - x_3, y_2 - y_3))$$

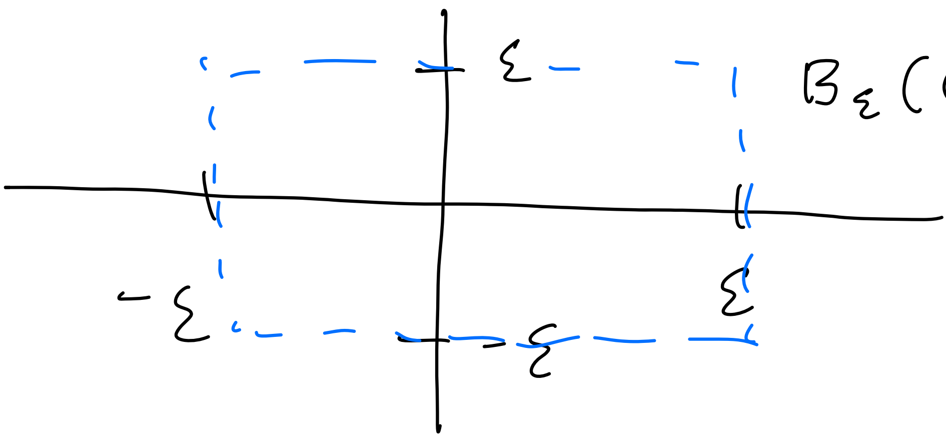
$$\leq d((x_1 - x_3, y_1 - y_3), (0,0)) + d((x_2 - x_3, y_2 - y_3), (0,0))$$

$$= d((x_1, y_1), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3)).$$

b) $B_\varepsilon((0,0))$

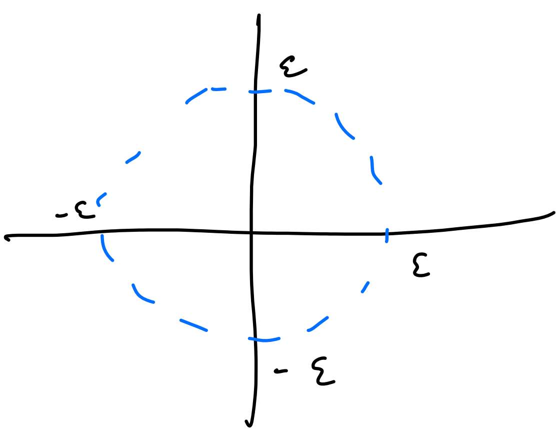
$$= \{(x,y) \in \mathbb{R}^2 \mid |x| < \varepsilon, \ |y| < \varepsilon\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid x \in (-\varepsilon, \varepsilon), \ y \in (-\varepsilon, \varepsilon)\}$$



$B_\varepsilon((0,0))$

d: $\{(x,y) \mid \sqrt{x^2 + y^2} < \varepsilon\}$



5)

Prove that a function $f : S \to S^*$ is continuous if and only if for ever closed set $E \subset S^*$, $f^{-1}(E)$ is also closed.

Note $x \in f^{-1}(S^* \backslash E)$ if and only if $f(x) \in S^* \backslash E$ if and only if $f(x) \notin E$. So $f^{-1}(S^* \backslash E) = S \backslash f^{-1}(E)$.

Suppose f is continuous. Let $E \subset S^*$ be closed. Then $S^* \backslash E$ is open, so $f^{-1}(S^* \backslash E)$ is open. So $f^{-1}(E)$ is closed.

Conversely, suppose that $f^{-1}(E)$ is closed whenever $E \subseteq S^*$ is closed. Then if E is open, $S^* \backslash E$ is closed, so $f^{-1}(S^* \backslash E)$ is closed and $f^{-1}(E)$ is open. Thus f is continuous.

b) a) Since inf and sup are lower and upper bounds respectively, if $s \in I$, $x \leq s \leq y$ so $s \in [x, y]$. Thus $I \subseteq [x, y]$.

Let $s \in (x, y)$. Since $x = \inf I$, $x < s$, there exists $a \in I$ such that $x \leq a < s$. Since $y = \sup I$, $s < y$, there exists $b \in I$ such that $y < b \leq y$. Thus $a < y < b$, and by the assumed property $y \in I$. So $(x, y) \subseteq I$.

b) There are sequences in I converging to x and y, so if I is closed, $x, y \in I$. Thus $[x, y] = \{x, y\} \cup (x, y) \subseteq I \subseteq [x, y]$, so $I = [x, y]$.

c) $(x - \frac{1}{n}), (y + \frac{1}{n})$ are sequences in $\mathbb{R} \setminus I$ converging to x, y, so if I is open, $\mathbb{R} \setminus I$ is closed, and $x, y \in \mathbb{R} \setminus I$.

Since $I \subseteq [x, y]$, we conclude that $I \subseteq (x, y)$ and since $(x, y) \subseteq I$, $I = (x, y)$.

7) Let S be a metric space with distance function d. Let E and F be connected subsets of S such that $E \cap F$ is nonempty. Prove that $E \cup F$ is connected.

Suppose not. Let A, B be open sets in S such that $A \cap B \cap (E \cup F) = \varnothing$, $E \cup F \subseteq A \cup B$, $A \cap (E \cup F) \neq \varnothing$ and $B \cap (E \cup F) \neq \varnothing$. We can conclude that

$$A \cap B \cap E = \varnothing$$

$$A \cap B \cap F = \varnothing$$

$$E \subseteq A \cup B$$

$$F \subseteq A \cup B.$$

Since E and F intersect, and $E \cup F \subset A \cup B$, a point of the intersection is in either A or B, so either A or B intersects both E and F. It follws that either E or F intersects both A and B. Then either E or F satisfies all the conditions for being disconnected. This is a contradiction.