Homework 8

Due Monday, October 31 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. a) Prove that $f : [1, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous.
   b) Prove that $f : [0, 2] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous (Hint: Use the result of HW 6, problem 2).
   c) Use parts a and b to prove that $f : [0, \infty)$, $f(x) = \sqrt{x}$ in uniformly continuous.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.

3. Let S be a metric space with distance function d. Let $x \in S$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$. Prove that $B_\epsilon(x) = \{y \in S | d(y, x) < \epsilon\}$ is an open set. Prove that $\overline{B}_\epsilon(x) = \{y \in S | d(x, y) \le \epsilon\}$ is a closed set.

4. a) Prove that $\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$ is a metric space with the alternate distance function
   $$d'((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$
   (Hint: for the triangle inequality, first proove that
   $$d'((x_1, y_1), (x_2, y_2)) \le d'((x_1, y_1), (0, 0)) + d'((x_2, y_2), (0, 0)).$$
   Then prove
   $$d'((x_1, y_1), (x_2, y_2)) = d'((x_1 - a, y_1 - b), (x_2 - a, y_2 - b))$$
   for any $a, b \in \mathbb{R}$.)
   b) Let $\epsilon > 0$. Draw a picture of the set $B_\epsilon((0, 0)) = \{(x, y) \in \mathbb{R}^2 | d'((x, y), (0, 0)) < \epsilon\}$, and the set $\{(x, y) \in \mathbb{R}^2 | d((x, y), (0, 0)) < \epsilon\}$, where d is the usual distance function
   $$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$
   c) **If you are interested, no need to hand in:** Prove the triangle inequality for the usual distance function d on $\mathbb{R}^2$ as given above.

5. Let S be a metric space with distance function d. Let S^* be a metric space with distance function d^*. Prove that a function $f : S \to S^*$ is continuous if and only if for ever closed set $E \subset S^*$, $f^{-1}(E)$ is also closed.

6. Let $I \subset \mathbb{R}$ be a bounded nonempty set with the property that if $a, b \in I$ and $a < c < b$ then $c \in I$. Let $x = \inf I$ and $y = \sup I$.
   a) Prove that $(x, y) \subseteq I \subseteq [x, y]$.
   b) If I is closed prove that $I = [x, y]$.
   c) If I is open prove that $I = (x, y)$.

7. Let S be a metric space with distance function d. Let E and F be connected subsets of S such that $E \cap F$ is nonempty. Prove that $E \cup F$ is connected.