Homework 7

1. Use the $\epsilon - \delta$ property to show that the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous (i.e. for each $x_0 \in \mathbb{R}$, given $\epsilon > 0$, find $\delta > 0$ such that$|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$):

   a) $f(x) = x^2$

   b) $f(x) = x^3$. (Hint: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$)

a)

Let $x_0 \in \mathbb{R}$. Let $\epsilon > 0$. Chose $\delta = \min\{1, \frac{\epsilon}{2|x_0|+1}\}$. Then if $|x - x_0| < \delta$,

$$|x^2 - x_0^2| = |x - x_0||x + x_0| < \delta(2|x_0| + \delta) \leq \epsilon.$$

b)

Let $x_0 \in \mathbb{R}$. Let $\epsilon > 0$. Choose $\delta = \min\{1, \frac{\epsilon}{3|x_0|^2+3|x_0|+1}\}$. If $|x - x_0| < \delta$ then

$$|x^3 - x_0^3| = |x - x_0||x^2 + xx_0 + x_0^2| < \delta((|x_0| + \delta)^2 + |x_0|(|x_0| + \delta) + |x_0|^2)$$

$$\leq \delta((|x_0| + 1)^2 + |x_0|(|x_0| + 1) + |x_0|^2) \leq \epsilon.$$

2. Prove that $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$, is continuous.

If $x_0 = 0$ let $\delta = \epsilon^2$. Then if $x \in [0, \infty)$ and $|x| < \delta$, $|f(x)| = \sqrt{x} < \sqrt{\delta} = \epsilon$.
If $x_0 > 0$, chose $\delta = \sqrt{x_0}\epsilon$. Then if $x \in [0, \infty]$ and $|x - x_0| < \delta$, multiplying and dividing by $\sqrt{x_0} + \sqrt{x}$ we have

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{|\sqrt{x} + \sqrt{x_0}|} < \frac{\delta}{\sqrt{x_0}} = \epsilon$$

3. In each part, prove that $f : \mathbb{R} \to \mathbb{R}$ is is not continuous at $x_0 = 0$.

   a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.
   b) $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

   a) $\frac{1}{n} \to 0$ but $f(\frac{1}{n}) = 1$ does not converge to $f(0) = 0$.

   b) $\frac{2}{\pi(2n+1)} \to 0$ but $f(\frac{2}{\pi(2n+1)}) = \sin(\frac{(2n+1)\pi}{2}) = 1$ does not converge to $f(0) = 0$.

4) Let $\varepsilon = f(x_0) > 0$. There exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$, and so

$$f(x) > f(x_0) - \varepsilon = 0.$$

Thus $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

5) Ross 17.12 Hint: use the density of the rationals $\mathbb{Q}$

a) Assume towards a contradiction that $f(x_0) \neq 0$ for $x \in (a,b)$. Then there exists $\delta > 0$ such that $x \in (a,b)$, $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < |f(x_0)|$ and thus $|f(x)| > 0$. By denseness of $\mathbb{Q}$, there exists $r \in \mathbb{Q} \cap (a,b)$, $x_0 - \delta < r < x_0 + \delta$. Then $|r - x_0| < \delta$ but $f(r) = 0$, a contradiction.

b) $f(x) - g(x)$ has the property above and thus is 0.

6) Ross 17.13 Hint: also use the density of the irrationals $\mathbb{R}\backslash\mathbb{Q}$

a) Let $r \in \mathbb{Q}$. For all n, let s_n be an irrational number in $(r, r+\frac{1}{n})$. Then $s_n \to r$ but $f(s_n) = 0$ does not converge to $f(r) = 1$. So f is not continuous at r.

   Let $s \in \mathbb{R}\backslash\mathbb{Q}$. For all n, let s_n be a rational number in $(s, s+\frac{1}{n})$. Then $s_n \to s$ but $f(s_n) = 1$ does not converge to $f(r) = 0$.

b Let $x \neq 0$, chose $\epsilon = |x|$. Let $\delta > 0$. There is a irrational number $s \in (x-\delta, x+\delta)$, but $|f(x) - f(s)| = |x| \geq \epsilon$.

   Let $\epsilon > 0$. Chose $\delta = \epsilon$. If $|x| < \delta$, either $f(x) = 0 < \epsilon$ or $|f(x)| = |x| < \epsilon$. Either way, this proves f is continuous.

7) Since E is not closed, there exists a sequence (s_n) in E such that $s_n \to s$ and $s \notin E$.

Let $f: E \to \mathbb{R}$ be defined by

$f(x) = \dfrac{1}{x-s}$; this is not a problem since $s \notin E$. f is continuous since it is a rational function.

We show that f is not bounded.

Let $M > 0$; choose $\varepsilon = \frac{1}{M}$. There exists $n \in \mathbb{N}$ such that $|s_n - s| < \varepsilon$, so

$|f(s_n)| = \dfrac{1}{|s_n - s|} > \dfrac{1}{\varepsilon} = M$. So

$f(E) \nsubseteq [-M, M]$ for any $M > 0$, and f is not bounded.

8) Assume $\lim_{x \to x_0} f(x) = L$.

Let (s_n) be a sequence in (a, x_0), $s_n \to x_0$. Then (s_n) is a sequence in $(a,b) \setminus \{x_0\}$, so $g(s_n) = f(s_n) \to L$. Thus $\lim_{x \to x_0} g(x) = L$.

Let (t_n) be a sequence in (x_0, b), $t_n \to x_0$. Then (t_n) is a sequence in $(a,b) \setminus \{x_0\}$, so $h(t_n) = f(t_n) \to L$. Thus $\lim_{x \to x_0} h(x) = L$.

Now assume $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L$.

Let $\varepsilon > 0$. There exist $\delta_1, \delta_2 > 0$ such that

If $x \in (a, x_0)$ and $|x - x_0| < \delta_1$ then $|g(x) - L| < \varepsilon$
so $|f(x) - L| < \varepsilon$

If $x \in (x_0, b)$ and $|x - x_0| < \delta_2$ then $|h(x) - L| < \varepsilon$
so $|f(x) - L| < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. If $x \in (a,b) \setminus \{x_0\}$ then either $x \in (a, x_0)$, $x \in (x_0, b)$, and if $|x - x_0| < \delta \le \delta_1, \delta_2$ then in either case $|f(x) - L| < \varepsilon$. So

$\lim_{x \to x_0} f(x) = L$.