Homework 7

Due Monday, October 24 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Use the $\epsilon - \delta$ property to show that the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous (i.e. for each $x_0 \in \mathbb{R}$, given $\epsilon > 0$, find $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$):
   a) $f(x) = x^2$
   b) $f(x) = x^3$. (Hint: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$)

2. Prove that $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$, is continuous.

3. In each part, prove that $f : \mathbb{R} \to \mathbb{R}$ is is not continuous at $x_0 = 0$.
   a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.
   b) $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Prove that there is an open interval $(a, b) \subseteq \mathbb{R}$ such that $x_0 \in (a, b)$ and $f(x) > 0$ for all $x \in (a, b)$.

5. Ross 17.12 Hint: use the density of the rationals $\mathbb{Q}$

6. Ross 17.13 Hint: also use the density of the irrationals $\mathbb{R}\backslash\mathbb{Q}$

7. Let $E \subset \mathbb{R}$ be a set which is not closed. Show that there exists $f : E \to \mathbb{R}$ such that f is continuous and $f(E)$ is not bounded. (Hint: The function should have the form $1/(x - c)$. What should c be?)

8. Let $a, b, x_0 \in \mathbb{R}$, $a < x_0 < b$, and let $f : (a, b) \to \mathbb{R}$ be a function. Define
$$g : (a, x_0) \to \mathbb{R} \quad g(x) = f(x) \text{ for all } x \in (a, x_0)$$
$$h : (x_0, b) \to \mathbb{R} \quad h(x) = f(x) \text{ for all } x \in (x_0, b).$$
(that is, g, h are simpy the restrictions of f to smaller domains). Prove that $\lim_{x \to x_0} f(x) = L$ if and only if
$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L.$$
(This may be a familiar theorem from calculus about left and right limits.)

Extra (not graded): Give an example of functions $f, g : \mathbb{R} \to \mathbb{R}$, such that
$$\lim_{x \to x_0} f(x) = L, \quad \lim_{x \to L} g(x) = M \text{ but } \quad \lim_{x \to x_0} g \circ f(x) \neq M.$$