1)

a) $E \subseteq \mathbb{R}$ is open if for each $s \in E$ there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \subset E$.

(OR, if $\mathbb{R} \setminus E$ is closed)

b) $E \subseteq \mathbb{R}$ is closed if for each sequence (s_n) in E, $s_n \to s$ implies $s \in E$

(OR, if $\mathbb{R} \setminus E$ is open)

c) $(0, 1)$

d) $[0, 1]$

e) $[0, 1)$

f) $\mathbb{R}$

2) If $n < 2^k$, $\frac{1}{n} > \frac{1}{2^k}$ so

$$\sum_{n=2^{k-1}}^{2^k-1} \frac{1}{n} > \sum_{n=2^{k-1}}^{2^k-1} \frac{1}{2^k} = \frac{2^k-1-(2^{k-1}-1)}{2^k} \leftarrow \text{\# of terms in sum}$$

$$= \frac{1}{2}$$

Let $S_m = \sum_{n=1}^{m} \frac{1}{n}$. Then

$$S_{2^N-1} = \sum_{k=1}^{N} \sum_{n=2^{k-1}}^{2^k-1} \frac{1}{n} > \sum_{k=1}^{N} \frac{1}{2} = \frac{N}{2}$$

Thus S_m is unbounded and does not converge

3) Ross 15.1

a) Since $\frac{1}{n} > \frac{1}{n+1} > 0$ and $\frac{1}{n} \to 0$,

$\sum (-1)^n \frac{1}{n}$ converges by the alternating series test.

b)

$$\frac{n!}{2^n} = \frac{(n)(n-1)\dots(4)(3)(2)(1)}{(2)(2)\dots(2)(2)(2)(2)} > 1$$ as long as $n \geq 4$.

So $\left|\frac{(-1)^n n!}{2^n}\right| = \frac{n!}{2^n}$ does not converge to 0,

and thus $\frac{(-1)^n n!}{2^n}$ does not converge to 0

and $\sum \frac{(-1)^n n!}{2^n}$ does not converge.

Ross 15.2

a) $\sin\left(\frac{(12K+3)\pi}{6}\right)^n = \sin\left(\frac{\pi}{2}\right)^n = 1$

this subsequence of $\left(\sin\left(\frac{n\pi}{6}\right)^n\right)$ converges to 1

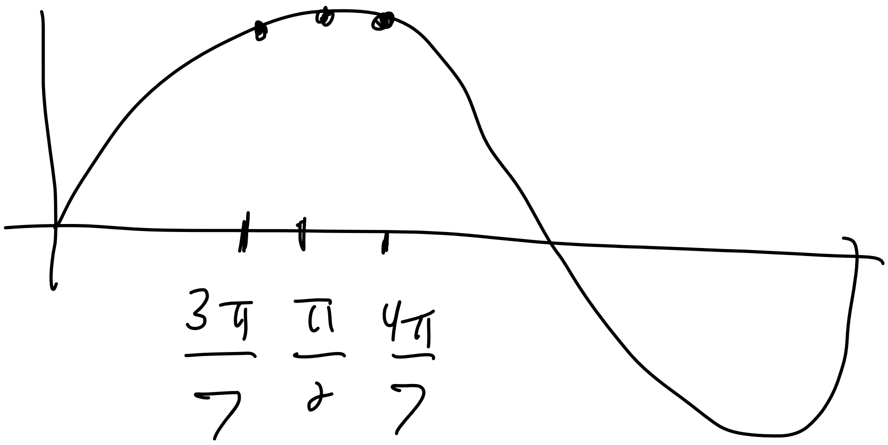
So $\sin\left(\frac{n\pi}{6}\right)^n$ does not converge to 0 and

$\sum \sin\left(\frac{n\pi}{6}\right)^n$ does not converge b)

$$\max \left\{ \left|\sin\left(\frac{n\pi}{7}\right)\right| \;\middle|\; n \in \mathbb{N} \right\}$$

$$= \max \left\{ \left|\sin\left(\frac{\pi}{7}\right)\right|, \left|\sin\left(\frac{2\pi}{7}\right)\right|, \dots \left|\sin\left(\frac{14\pi}{7}\right)\right| \right\}$$

$$= \sin\left(\frac{3\pi}{7}\right) < 1$$



So $\left|\sin\left(\frac{n\pi}{7}\right)^n\right| \leq \sin\left(\frac{3\pi}{7}\right)^n$

and $\sum \sin\left(\frac{3\pi}{7}\right)^n$ is a convergent geometric series, so $\sum \sin\left(\frac{n\pi}{7}\right)^n$ converges.

4)

a) Let $s \in (1,\infty)$ and $\varepsilon = s-1 > 0$.

Then $(s-\varepsilon, s+\varepsilon) = (1, 2s-1) \subset (1,\infty)$.

So $(1,\infty)$ is open

OR

Let (s_n) be a sequence in $\mathbb{R} \setminus (1,\infty) = (-\infty, 1]$ such that $s_n \to s$. Then $s_n \le 1$, so $s \le 1$, and $s \in (-\infty, 1]$. Thus $\mathbb{R} \setminus (1,\infty)$ is closed, so $(1,\infty)$ is open.

Define $s_n = 1 + 1/n$. Then $s_n \in (1,\infty)$ but $s_n \to 1+0 = 1 \notin (1,\infty)$. So $(1,\infty)$ is not closed.

OR

$1 \in \mathbb{R} \setminus (1,\infty)$, but for all $\varepsilon > 0$, $(1-\varepsilon, 1+\varepsilon)$ contains $1+\frac{\varepsilon}{2} \in (1,\infty)$. So $(1-\varepsilon, 1+\varepsilon) \not\subset \mathbb{R} \setminus (1,\infty)$ and $\mathbb{R} \setminus (1,\infty)$ is not open. Thus $(1,\infty)$ is not closed.

b) Let (s_n) be a sequence in $[1,\infty)$, such that $s_n \to s$. Since $s_n \geq 1$, $s \geq 1$, and $s \in [1,\infty)$. So $[1,\infty)$ is closed.

OR,

Let $s \in \mathbb{R} \setminus [1,\infty) = (-\infty, 1)$ and let $\varepsilon = 1 - s > 0$. Then $(s-\varepsilon, s+\varepsilon) = (2s-1, 1) \subset (-\infty, 1)$. So $\mathbb{R} \setminus [1,\infty)$ is open and $[1,\infty)$ is closed.

$1 \in [1,\infty)$, but for all $\varepsilon > 0$, $(1-\varepsilon, 1+\varepsilon)$ contains $1 - \frac{\varepsilon}{2} < 1$, so $(1-\varepsilon, 1+\varepsilon) \not\subset [1,\infty)$. So $[1,\infty)$ is not open.

OR,

$(s_n = 1 - \frac{1}{n})$ is a sequence in $\mathbb{R} \setminus [1,\infty) = (-\infty, 1)$, but $s_n \to 1 \notin (-\infty, 1)$, so $(-\infty, 1)$ is not closed and $[1,\infty)$ is not open

c) $[0,1)$ is not closed because $(1-\frac{1}{n})$ is a sequence in $[0,1)$ converging to $1 \notin [0,1)$.

$[0,1)$ is not open because $(-\frac{1}{n})$ is a sequence in $\mathbb{R} \setminus [0,1) = (-\infty, 0) \cup [1, \infty)$ converging to $0 \in [0,1)$. So $\mathbb{R} \setminus [0,1)$ is not closed and $[0,1)$ is not open.

d) Let $E = \{0\} \cup \{1, 1/2, 1/3, ..., 1/n, ...\}$.

Let $s \in \mathbb{R} \setminus E$.
If $s < 0$, and $|s-t| < -s$, then $t < s - s = 0$. Thus setting $r = -s$, $\{t \in \mathbb{R} | |s-t| < r\} \subset \mathbb{R} \setminus E$.
If $s > 1$, and $|s-t| < s-1$, then $t > s-(s-1) = 1$. Thus setting $r = s-1$, $\{t \in \mathbb{R} | |s-t| < r\} \subset \mathbb{R} \setminus E$.
Otherwise, $0 < s < 1$ and $s \neq \frac{1}{n}$ for all $n \in \mathbb{N}$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{N+1} < s < \frac{1}{N}$. Let $r = \min\{s - \frac{1}{N+1}, \frac{1}{N} - s\}$. If $|s-t| < r$ then $t > s-r \geq s-(s-\frac{1}{N+1}) = \frac{1}{N+1}$ and $t < s+r \leq s + \frac{1}{N} - s = \frac{1}{N}$. Thus $t \neq \frac{1}{n}$ for all $n \in \mathbb{N}$ and $t \notin E$. Thus setting $\{t \in \mathbb{R} | |s-t| < r\} \subset \mathbb{R} \setminus E$.
This proves that $\mathbb{R} \setminus E$ is open and E is closed.

Alternatively: $\mathbb{R}\backslash E = (-\infty, 0) \cup (1, \infty) \bigcup_{n\in\mathbb{N}} \left(\frac{1}{N+1}, \frac{1}{N}\right)$. Since each interval is open problem 6 proves that $\mathbb{R}\backslash E$ is open.

Alternatively: (this is just a sketch, fill in the details) Let (s_n) be a convergent sequence in E. If the set $\{s_n\}$ is finite, (s_n) must eventually become a constant sequence in order to coverge, and thus converges to an element of E. If $\{s_n\}$ is infinite, the elements of (s_n) must be getting smaller and smaller, and thus (s_n) must converge to $0 \in E$.

For any $r > 0$, the set $\{s \in \mathbb{R} | |s - 0| < r\}$ contains negative numbers, and therefore is not contained in E. This proves that E is not open.

e) Let $x \in \mathbb{R}\backslash\mathbb{Q}$. For any $r > 0$, $(x - r, x + r)$ contains a rational number, by denseness of $\mathbb{Q}$. So $\{y \in \mathbb{R} | |x - y| < r\} = (x - r, x + r)$ is not contained in $\mathbb{R}\backslash\mathbb{Q}$. So $\mathbb{R}\backslash\mathbb{Q}$ is not closed. A nearly identical argument, using denseness of the irrationals, implies $\mathbb{Q}$ is not open.

Alternatively, in HW 3 we found a sequence in $\mathbb{Q}$ converging to an irraitonal number. Thus $\mathbb{Q}$ is not closed. $\frac{\sqrt{2}}{n} \to 0$ is a sequence of irrational numbers converging to a rational number, so $\mathbb{R}\backslash\mathbb{Q}$ is not closed, and $\mathbb{Q}$ is not open.

5) Since E is open and $s \in E$, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \subseteq E$. Since $s_n \to s$, there exists $N \in \mathbb{R}$ such that $n > N$ implies $|s_n - s| < \varepsilon$, which in turn implies $s_n \in (s-\varepsilon, s+\varepsilon) \subseteq E$.

b) Since E is sequentially compact, it is bounded.

Since E is nonempty and bounded, $\sup E$ and $\inf E$ exist by the completeness axiom.

We have shown that there exists a sequence (s_n) in E such that $s_n \to \sup E$ and (t_n) such that $t_n \to \inf E$.

Thus since E is closed, $\sup E, \inf E \in E$.

OR

Suppose $\sup E \notin E$. For all $\varepsilon > 0$, there exists $s \in E$ such that $\sup E - \varepsilon < s < \sup E$,

So $s \in (\sup E - \varepsilon, \sup E + \varepsilon) \not\subset \mathbb{R} \setminus E$.

Thus $\mathbb{R} \setminus E$ is not open, contradicting the fact that E is closed.

7)

a) Let $s \in \bigcup_{n=1}^{\infty} U_n$. Then $s \in U_n$ for some $n_0 \in \mathbb{N}$. Since U_{n_0} is open, there exists $\varepsilon > 0$ such that

$$(s-\varepsilon, s+\varepsilon) \subset U_{n_0} \subset \bigcup_{n=1}^{\infty} U_n$$

b) Let (s_k) be a sequence in $\bigcap_{n=1}^{\infty} V_n$. such that $s_k \to s$.

For each $n_0 \in \mathbb{N}$, (s_k) is a sequence in $\bigcap_{n=1}^{\infty} V_n \subseteq V_{n_0}$, and V_k is closed, so $s \in V_{n_0}$. Since this is true for all $n_0 \in \mathbb{N}$, $s \in \bigcap_{n=1}^{\infty} V_n$.

c) We use induction on $n \in \mathbb{N}$. If $n=1$, U_1 is open by definition.

Now assume $E = U_1 \cap \dots \cap U_{n-1}$ is open.

Let $s \in E \cap U_n$. Since $s \in E$ and E is open, there exists $\varepsilon_1 > 0$ such that $(s - \varepsilon_1, s + \varepsilon_1) \subseteq E$.

Since $s \in U_n$ and U_n is open, there exists $\varepsilon_2 > 0$ such that $(s - \varepsilon_2, s + \varepsilon_2) \subseteq U_n$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, which is > 0.

Then $(s - \varepsilon, s + \varepsilon) \subseteq (s - \varepsilon_1, s + \varepsilon_1) \subseteq E$ and

$\subseteq (s - \varepsilon_2, s + \varepsilon_2) \subseteq U_n$

So $(s - \varepsilon, s + \varepsilon) \subset E \cap U_n$.

d) $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = (-1, 1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{1}{3}, \frac{1}{3}\right) \ldots$

$= \{0\}$ is not open

e $\displaystyle\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \ldots$

$= (0, 1]$ is not closed.