# Homework 6

Due Monday, October 10 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. a) Give a definition of an open subset of $\mathbb{R}$.
   b) Give a definition of a closed subset of $\mathbb{R}$
   c) Given an example of a subset of $\mathbb{R}$ which is open and not closed.
   d) Given an example of a subset of $\mathbb{R}$ which is closed and not open.
   e) Given an example of a subset of $\mathbb{R}$ which is neither closed nor open.
   f) Give and example of a subset of $\mathbb{R}$ which is both open and and closed.

2. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. See Ross 14.14; the method there can be rephrased as:

$$\text{Compare } \sum_{n=2^{(k-1)}}^{2^k-1} \frac{1}{n} \text{ to } \sum_{n=2^{(k-1)}}^{2^k-1} \frac{1}{2^k}$$

3. Ross 15.1 and 15.2.

4. Prove that

   a) $(1, \infty)$ is open and not closed.
   b) $[1, \infty)$ is closed and not open.
   c) $[0, 1)$ is neither open nor closed
   d) $\{0\} \cup \{1, 1/2, 1/3, ..., 1/n, ...\}$ is closed and not open.
   e) $\mathbb{Q}$ is neither open nor closed.

5. Let $E \subset \mathbb{R}$ be an open set. Let $(s_n)$ be a sequence which converges to $s \in E$. Prove that there exists $N \in \mathbb{R}$ such that $n > N$ implies $s_n \in E$.

6. Let $E \subset \mathbb{R}$ be nonempty and sequentially compact. Prove that $\sup E$, $\inf E$ exist, and that $\sup E \in E$ and $\inf E \in E$.

7. For each $n \in \mathbb{N}$, let $U_n$ be an open set and let $V_n$ be a closed set.

   a) Prove that the union of all the $U_n$ is open.
   b) Prove the the intersection of all the $V_n$ is closed.
   c) Prove that for any $n \in \mathbb{N}$, $U_1 \cap U_2 ... \cap U_n$ is open.
   d) Give an example of collection of open sets $U_n$ such that the intersection of all the $U_n$ is not open.
   e) Give an example of collection of closed sets $V_n$ such that the union of all the $V_n$ is not closed.