1) 9,12

a) Assume $\left|\frac{s_{n+1}}{s_n}\right| \to L < 1.$

Let $a = \frac{1+L}{2}$, so $0 \leq L < a < 1$. Then

there exists $N \in \mathbb{N}$ such that

for $n \geq N$ $\left|\frac{s_{n+1}}{s_n}\right| < a$.

Thus for $n > N$,

$$|s_n| = \left|\frac{s_n}{s_{n-1}}\right|\left|\frac{s_{n-1}}{s_{n-2}}\right| \cdots \left|\frac{s_{n-(n-N-1)}}{s_N}\right| |s_N|$$

$$< a^{n-N} |s_N|$$

Since $|a| < 1$,

$$\lim_{n\to\infty} a^{n-N}|s_N| = \frac{|s_N|}{a^N} \lim_{n\to\infty} a^n = 0$$

So $0 \leq |s_n| \leq a^{n-N}|s_N|$ for $n > N$

implies $|s_n| \to 0$.

b) If $\left|\frac{s_{n+1}}{s_n}\right| \to L > 1$, let $t_n = \frac{1}{|s_n|}$. Then

$\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{s_n}{s_{n+1}}\right| = \frac{1}{\left|\frac{s_{n+1}}{s_n}\right|} \to \frac{1}{L} < 1$, so $t_n \to 0$ and

$|s_n| \to \infty$ by a theorem in class (Also problem 4 c)

9.14

Let $s_n = \frac{a^n}{n^p}$. Then $\left|\frac{s_{n+1}}{s_n}\right| = |a|\left(\frac{n}{n+1}\right)^p \to |a|(1)^p$
$= |a|.$

By 9.12, if $|a| < 1$, $\frac{a^n}{n^p} \to 0.$

If $|a| > 1$ $|s_n| \to \infty$. If $a > 1$, $s_n = |s_n|$.

If $a < -1$, $s_{2n} = |s_{2n}| \to \infty$ while

$s_{2n+1} = -|s_{2n+1}| \to -\infty.$

If $a = 1$, $\frac{1}{n^p} \to 0$.

2) 11.10

The set of subseq. limits is
$\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\} = \{0\} \cup \{\frac{1}{k} \mid k \in \mathbb{N}\}$.

Let $\varepsilon > 0$.

0/ choose $N > \frac{1}{\varepsilon}$. Then $\{\frac{1}{n} \mid n > N, n \in \mathbb{N}\} \subset (-\varepsilon, \varepsilon)$
since $\{\frac{1}{n} \mid n \in \mathbb{N}, n > N\}$ is infinite and contained in the set of values of (s_n),
$\{n \in \mathbb{N} \mid |s_n - 0| < \varepsilon\}$ is infinite. Thus 0 is a S.L.

$\frac{1}{k}$/ Let $k \in \mathbb{N}$

$$\{n \in \mathbb{N} \mid s_n = \frac{1}{k}\} = \{n \in \mathbb{N} \mid |s_n - \frac{1}{k}| = 0\}$$
$$\subseteq \{n \in \mathbb{N} \mid |s_n - \frac{1}{k}| < \varepsilon\}$$

is infinite. So $\frac{1}{k}$ is an SL.

Now assume $r \neq 0, \frac{1}{k}$ for any $k \in \mathbb{N}$.
If $r < 0$, $\{n \in \mathbb{N} \mid |s_n - r| < r\} = \{n \in \mathbb{N} \mid 2r < s_n < 0\}$
is empty, so r is not a S.L.
If $r > 1$, $\{n \in \mathbb{N} \mid |s_n - r| < r-1\} = \{n \in \mathbb{N} \mid 1 < s_n < 2r-1\}$
is empty, so r is not a S.L.

Pf $0<r<1$, Let $K = \max\{n \in \mathbb{N} \mid n \leq \frac{1}{r}\}$, which is finite and nonempty (1 is in it). Then $K \leq \frac{1}{r} < K+1$, and since $r \neq \frac{1}{K}$,

$$\frac{1}{K+1} < r < \frac{1}{K}$$

3) Ross 12.6

There is a proof straight from def. This uses subsequences.

a) Use c instead of k

(t_k) is a convergent subsequence of (s_n) with $\lim(t_k) = a$

if and only if

(ct_k) is a convergent subsequence of (s_n) with $\lim(ct_k) = ca$

So $\{b \mid b$ is an S.L. of $(cs_n)\} = \{ca \mid a$ is an S.L. of $(s_n)\}$

Since $c > 0$, the max of this set is $c \limsup(s_n) = \limsup(cs_n)$

b) Identical proof with <u>min</u> at the end.

c) Identical proof but $\limsup(cs_n) = \max\{$S.L.'s of $(cs_n)\}$

$= \max\{c \times$ SL's of $(s_n)\}$

$= c \times \min\{$SL's of $(s_n)\}$.

$= c \liminf(s_n)$

etc.

12.8

Let $V_N = \sup\{s_n \mid n > N\}$ $V_N' = \sup\{t_n \mid n > N\}$ $V_N'' = \sup\{s_n t_n \mid n > N\}$

Then if $n > N$, $s_n t_n \leq V_N t_n \leq V_N V_N'$ since everything ≥ 0

so $V_N'' \leq V_N V_N'$ and

$\lim(V_N'' - V_N V_N') = \lim\sup(s_n t_n) - \lim\sup(s_n) \lim\sup(t_n) \leq 0.$

4) a) If (s_n) were decreasing, it would be bounded above by s_1. So (s_n) is increasing.

Let $M \in \mathbb{R}$. Since (s_n) is not bounded above, there exists $N \in \mathbb{N}$ such that $s_N > M$. Since (s_n) is increasing, if $n > N$, $s_n \geq s_N > M$. So $s_n \to \infty$.

b) Let $K \in \mathbb{N}$. Assume we have chosen $t_1, \ldots, t_K$ s.t. $t_K = s_{n_K}$, $n_1 < \ldots < n_K$ and $t_K > K$. Since (s_n) is not bounded, $\{n \mid s_n > K+1\}$ is infinite, and we can choose $n_{K+1} > n_K$ such that $s_{n_{K+1}} > K+1$. We can define a sequence (t_K) inductively in this way.

Let $M \in \mathbb{R}$. Choose $N = M$. If $K > N$, then $t_K = s_{n_K} > K > M$. So $t_K \to \infty$.

c) Assume $s_n \to 0$. Let $M \in \mathbb{R}$. There exists $N \in \mathbb{R}$ such that $n > N$ implies $|s_n| < \frac{1}{|M|+1}$.
Thus $n > N$ implies $\frac{1}{|s_n|} > |M|+1 > M$. So $\frac{1}{|s_n|} \to \infty$.

Now assume $\frac{1}{|s_n|} \to \infty$. Let $\varepsilon > 0$. There exists $N \in \mathbb{R}$ such that $n > N$ implies $\frac{1}{|s_n|} > \frac{1}{\varepsilon}$.
Thus $n > N$ implies $|s_n| < \varepsilon$ and $s_n \to 0$.

5)

a) $\left(\dfrac{(k+1)^2}{3^{k+1}}\right)\left(\dfrac{3^k}{k^2}\right) = \dfrac{1}{3}\left(\dfrac{k+1}{k}\right)^2$

Since $\frac{k+1}{k} \to 1$, $\frac{1}{3}\left(\frac{k+1}{k}\right)^2 \to \frac{1}{3} < 1$.

b) $\left(\dfrac{k^2}{3^k}\right)^{1/k} = \dfrac{1}{3}\left(k^{1/k}\right)^2 \to \dfrac{1}{3} < 1.$

c) $\dfrac{k^2}{3^k} < \dfrac{1}{2^k}$ if $k^2 < \left(\frac{3}{2}\right)^k$

if

Since $k^{1/k} \to 1$, $k^{2/k} \to 1^2 = 1$, so

there exists $N \in \mathbb{R}$ such that

$k > N$ implies $k^{2/k} < 1 + \frac{1}{2} = \frac{3}{2}$.

Thus for $k > N$, $k^2 < \left(\frac{3}{2}\right)^k$

and $\dfrac{k^2}{3^k} < \dfrac{1}{2^k}$. So $\sum \frac{k^2}{3^k}$

converges by comparison to a geometric

series.

b) a) For $n \geq 2$, $n-1 \geq \frac{n}{2}$, and

$\frac{n-1}{n^2} \geq \frac{1}{2n}$. Since $\frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n}$ does not converge, $\sum_{n=2}^{\infty} \frac{n-1}{n^2}$ does not converge, and $\sum \frac{n-1}{n^2}$ does not converge.

b) $(-1)^n$ does not converge to 0, so $\sum (-1)^n$ does not converge.

c) $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2} = 3 \sum \frac{1}{n^2}$ converges (see chapter 15 - we did not discuss this)

d) $\frac{(n+1)^3}{3^{n+1}} \frac{3^n}{n^3} = \frac{1}{3} \left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$

$\sum \frac{n^3}{3^n}$ converges by Ratio test

e) $\frac{(n+1)^2}{(n+1)!} \frac{n!}{n^2} = \frac{1}{n+1} \left(1+\frac{1}{n}\right)^2 \to 0 < 1$

$\sum \frac{n^2}{n!}$ converges by ratio test.

7) Let $V_N = \sup\left\{ \frac{a_{n+1}}{a_n} \;\middle|\; n \in \mathbb{N},\ n \geq N \right\}$.

Since $V_N \to \limsup\left(\frac{a_{n+1}}{a_n}\right) < C$, choosing N large $\leftarrow \mathbb{N}$ enough $V_{N-1} < C$.

$\hat{\ }$ just to make indices work out.

Then if $n \geq N > N-1$, $\frac{a_{n+1}}{a_n} \leq V_{N-1} < C$.

So for $n \geq N$,

$$a_n = \left(\frac{a_n}{a_{n-1}}\right) \cdots \left(\frac{a_{N+1}}{a_N}\right) a_N$$

$$< C^{n-N} a_N.$$