Homework 5

Due Monday, October 3 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Ross 9.12 and 9.14

2. Ross 11.10

3. Ross 12.6 and 12.8

4. Prove the following theorems from lecture
   a) If (s_n) is monotonote and not bounded above, $s_n \to \infty$
   b) If (s_n) is not bounded above then then (s_n) has a subsequence which diverges to infinity
   c) If $s_n \neq 0$ for all $n \in \mathbb{N}$ then $s_n \to 0$ if and only if $\frac{1}{|s_n|} \to \infty$

5. Consider the series $\sum \frac{k^2}{3^k}$.
   a) Use the ratio test to show the series converges.
   b) Use the root test to show the series converges. (Recall that $\lim n^{1/n} = 1$)
   c) Use the limit comparison test to show the series converges (Hint: Compare to a geometric series. You might need to consider only large k.)

6. Ross 14.2

7. Let $a_n \in \mathbb{R}$, $a_n > 0$ for all $n \in \mathbb{N}$. Consider the sequence $\left(\frac{a_{n+1}}{a_n}\right)$. Let $C \in \mathbb{R}$ such that

$$\limsup \frac{a_{n+1}}{a_n} < C.$$

Prove that there exists $N \in \mathbb{N}$ such that for $n > N$,

$$a_n < C^{n-N} a_N.$$

Extra (Not graded): There exists a sequence (s_n) such that the set of values is

$$\{s_n | n \in \mathbb{N}\} = \mathbb{Q} \cap [0, 1].$$

Find the set of subsequence limits of (s_n).