1)

a) $\limsup = \lim_{N\to\infty} \sup\{s_n \mid n > N\}$

$\liminf = \lim_{N\to\infty} \inf\{s_n \mid n > N\}$

b) $(1, 2+\frac{1}{2}, 1, 2+\frac{1}{3}, 1, 2+\frac{1}{5}, \ldots,)$

c) $(\frac{1}{n})$ $\sup\{\frac{1}{n} \mid n \in \mathbb{N}\} = 1$

$\limsup \frac{1}{n} = \lim \frac{1}{n} = 0.$

2) a) Choose $\varepsilon = 1$. Let $N \in \mathbb{R}$. For $n+1, n > N$,

$|(n+1) - n| = 1 \geq \varepsilon$.

b) Choose $\varepsilon = 1$. Let $N \in \mathbb{R}$. For $2n+1, 2n > N$,

$|(-1)^{2n+1} - (-1)^{2n}| = 2 > \varepsilon$.

3) Let $t = \sup(S)$. If $t \in S$,

Let $s_n = t$ for all $n \in \mathbb{N}$.

Otherwise, assume $t \notin S$. Since

$t - 1 < t$, there exists $s_1 \in S$

such that $t - 1 < s_1 \leq t$, and

since $t \notin S$, $t - 1 < s_1 < t$.

Thus $\max\{t - \frac{1}{2}, s_1\} < t$, so there exists $s_2 \in S$

such that $\max\{t - \frac{1}{2}, s_1\} < s_2 < t$.

Continuing in this way, we choose s_n for each $n \in \mathbb{N}$

such that $\max\{t - \frac{1}{n}, s_{n-1}\} < s_n < t$.

Thus (s_n) is increasing, and $t-\frac{1}{n} < s_n < t$,
So $s_n \to t$ by the squeeze theorem

4) a) For all $N \in \mathbb{N}$, $T_N = \{1, -1\}$, so

$V_N = 1 = \limsup((-1)^n)$

$U_N = -1 = \liminf((-1)^n)$

If $s_n = (-1)^n$, $t_k = s_{2k+1} = (-1)^{2k+1} = -1$ is monotone (indeed, constant) and $t_k \to -1 = \liminf((s_n))$

b) $\frac{1}{n} \to 0$, so $\liminf(\frac{1}{n}) = 0$. $\frac{1}{n}$ is monotone, ($\frac{1}{n+1} < \frac{1}{n}$).

c) Let $s_n = (-1)^n + \frac{1}{n}$. Then $s_n \geq -1$, so $U_N \geq -1$ and $\liminf(s_n) \geq -1$. But

$t_k = s_{2k+1} = (-1)^{2k+1} + \frac{1}{2k+1} = -1 + \frac{1}{2k+1} \to -1$.

Since liminf is $\leq$ all subsequence limits, $\liminf(s_n) \leq -1$, so $\liminf(s_n) = -1$. t_k is monotone.

5) a) $\frac{2}{3}$, $\frac{5}{9}$, $\frac{14}{27}$

b) $S_1 > \frac{1}{2}$, if $S_n > \frac{1}{2}$ then $S_{n+1} = \frac{1}{3}(S_n+1) > \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$.

c) $S_{n+1} - S_n = \frac{1}{3} S_n + \frac{1}{3} - S_n = \frac{1}{3} - \frac{2}{3} S_n < \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{2} = 0$.

d) (S_n) is monotone and bounded (by $\frac{1}{2}$ and 1)

so converges. Since $\lim(S_{n+1}) = \lim(S_n)$

and $S_{n+1} = \frac{1}{3}(S_n + 1)$, $\lim(S_n) = \frac{1}{3} \lim(S_n) + \frac{1}{3}$

so $\lim(S_n) = \frac{1}{2}$.

b)

a)

$(1 + a + \dots + a^n)(1 - a)$

$= 1 + a + a^2 + \dots + a^{n-1} + a$

$\quad - a - a^2 - \dots - a^{n-1} - a - a^{n+1}$

$= 1 - a^{n+1}$

b) Let $\varepsilon > 0$. Choose N such that $n > N$ implies $\frac{1}{2^{n-2}} < \varepsilon$ (this is possible since $\frac{1}{2^n} \to 0$). For $m, n > N$, assume $m \geq n$, then

$|S_m - S_n| \leq |S_m - S_{m-1}| + \dots + |S_{n+1} - S_n|$

$\leq \frac{1}{2^{(m-1)}} + \dots + \frac{1}{2^n}$

$= \frac{1 - (\frac{1}{2})^m}{1 - (\frac{1}{2})} - \frac{1 - (\frac{1}{2})^n}{1 - (\frac{1}{2})}$

$= (\frac{1}{2})^{n-1} - (\frac{1}{2})^{m-1}$

$\leq 2(\frac{1}{2})^{n-1} = \frac{1}{2^{n-2}} < \varepsilon.$

7) Choose $S, T \in \mathbb{R}$ such that

$|s_n| \le S$, $|t_n| \le T$ for all $n \in \mathbb{N}$. Then

$|s_n + t_n| \le S + T$, so $(s_n + t_n)$ is bounded.

Define $V_N = \sup\{s_n \mid n \in \mathbb{N}, n \ge N\}$

$V_N' = \sup\{t_n \mid n \in \mathbb{N}, n \ge N\}$

$V_N'' = \sup\{s_n + t_n \mid n \in \mathbb{N}, n \ge N\}$.

If $n \in \mathbb{N}$, $n \ge N$, then $s_n + t_n \le V_N + V_N'$.

So $V_N + V_N'$ is an UB for $\{s_n + t_n \mid n \in \mathbb{N}, n \ge N\}$

and $V_N'' \le V_N + V_N'$ for all $N \in \mathbb{N}$.

Since $V_N'' - V_N - V_N' \le 0$ for all $N \in \mathbb{N}$,

$\lim(V_N'' - V_N - V_N') \le 0$ for all $N \in \mathbb{N}$,

So $\limsup(s_n + t_n) - \limsup(s_n) - \limsup(t_n) \le 0$.

The proof for liminf is identical with the inequalities reversed and V replaced with U.

Example: $s_n = (-1)^n$ $t_n = (-1)^{n+1}$ $s_n + t_n = (-1)^n(1-1) = 0$

so $\limsup(s_n + t_n) = 0 < \limsup((-1)^n) + \limsup((-1)^{n+1})$
$= 1 + 1 = 2$.