Homework 4

Due Monday, September 26 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. a) Give the definition of $\limsup$ and $\liminf$

   b) Give an example of a sequence (s_n) such that the set of values $\{s_n | n \in \mathbb{N}\}$ is infinite, $\limsup s_n = 2$, and $\liminf s_n = 1$.

   c) Given an example of a sequence (s_n) such that $\sup\{s_n | n \in \mathbb{N}\} > \limsup s_n$

2. Prove, directly from the definition of a Cauchy sequence, that the following sequences are not Cauchy:

   a) $s_n = n$

   b) $t_n = (-1)^n$

3. Let S be a bounded nonempty subset of $\mathbb{R}$. Prove that there is a increasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.

4. For each sequence, find a monotone subsequence converging to the $\liminf$ or show that none exists.

   a) $((-1)^n)$

   b) $\left(\frac{1}{n}\right)$

   c) $\left((-1)^n + \frac{1}{n}\right)$

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6. a) Let $a < 1$. Prove that $1 + a + ... + a^n = \frac{1-a^{n+1}}{1-a}$.

   b) Let (s_n) be a sequence in $\mathbb{R}$ such that $|s_{n+1} - s_n| < 1/2^n$ for all n. Prove that s_n is Cauchy.

7. Let $(s_n), (t_n)$ be bounded sequences. Prove that $(s_n + t_n)$ is bounded and

$$\limsup(s_n + t_n) \leq \limsup(s_n) + \limsup(t_n)$$

$$\liminf(s_n + t_n) \geq \liminf(s_n) + \liminf(t_n).$$

   Give examples of $(s_n), (t_n)$ such that the inequalities above are not equalitites.