1)

a) For all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $n > N$ implies $|s_n - s| < \varepsilon$.

b) There exists $\varepsilon > 0$ such that for all $N \in \mathbb{R}$ there exists $n > N$ such that $|s_n - s| \geq \varepsilon$.

c) $\left(-\frac{1}{n}\right)$

d) $\left((-1)^n\right)$

2) Ross 8.8

a) Let $\varepsilon > 0$. Choose $N = \frac{1}{\varepsilon}$. If $n > N$,

$$\left|\sqrt{n^2+1} - n - 0\right| = \left|\frac{(n^2+1) - n^2}{\sqrt{n^2+1} + n}\right| = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

b) Let $\varepsilon > 0$. Choose $N = \frac{1}{4\varepsilon}$ If $n > N$

$$\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| = \left|\frac{(n^2+n) - (n+\frac{1}{2})^2}{\sqrt{n^2+n} + n + \frac{1}{2}}\right| = \frac{1}{4}\left|\frac{1}{\sqrt{n^2+n}+n+\frac{1}{2}}\right|$$

$$< \frac{1}{4n} < \frac{1}{4N} = \varepsilon.$$

c) Let $\varepsilon > 0$. Choose $N = \frac{1}{32\varepsilon}$. If $n > N$,

$$\left|\sqrt{4n^2+n} - 2n - \frac{1}{4}\right| = \left|\frac{4n^2+n - (2n+\frac{1}{4})^2}{\sqrt{4n^2+n} + 2n + \frac{1}{4}}\right|$$

$$= \frac{1}{16\left(\sqrt{4n^2+n} + 2n + \frac{1}{4}\right)} < \frac{1}{32n} < \frac{1}{32N} = \varepsilon.$$

3)

a) Let $s \in \mathbb{R}$. Choose $\varepsilon = 1$. Let $N \in \mathbb{R}$.
Pick $n > \max\{-s+1, N\}$ Then
$|s_n - s| \geq s_n - s = n - s \geq s+1-s = 1 = \varepsilon$.

b) Let $s \in \mathbb{R}$. Assume, towards a contradiction,
that $\cos(\frac{n\pi}{3}) \to s$. Then there exists $N \in \mathbb{R}$
such that $n > N$ implies $|\cos(\frac{n\pi}{3}) - s| < 1$.
Pick $n \in \mathbb{N}$, $n > N$. Then $6n$, $6n+3 > N$ so

$$2 = |1 - (-1)| = \left|\cos\left(\frac{(6n)\pi}{3}\right) - \cos\left(\frac{(6n+3)\pi}{3}\right)\right|$$
$$\leq \left|\cos\left(\frac{(6n)\pi}{3}\right) - s\right| + \left|s - \cos\left(\frac{(6n+3)\pi}{3}\right)\right|$$
$$< 1 + 1 = 2,$$ a contradiction.

c) Let $s \in \mathbb{R}$. Assume, towards a contradiction,
that $\sin(\frac{n\pi}{3}) \to s$. Then there exists $N \in \mathbb{R}$
such that $n > N$ implies $|\sin(\frac{n\pi}{3}) - s| < \frac{1}{4}$.
Pick $n \in \mathbb{N}$, $n > N$. Then $3n$, $6n+1 > N$ so

$$\frac{\sqrt{3}}{2} = \left| 0 - \frac{\sqrt{3}}{2} \right| = \left| \sin\left(\frac{(3n)\pi}{3}\right) - \sin\left(\frac{(6n+1)\pi}{3}\right) \right|$$

$$\leq \left| \sin\left(\frac{(3n)\pi}{3}\right) - s \right| + \left| s - \sin\left(\frac{(3n+1)\pi}{3}\right) \right|$$

$< \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, a contradiction.

d) Let $s \in \mathbb{R}$. Assume $1 + (-1)^n \to s$. There exists $N \in \mathbb{R}$ such that $n > N$ implies $|1 + (-1)^n - s| < 1$. Pick $n \in \mathbb{N}$, $n > N$. Then $2n, 2n+1 > N$, so

$$2 = |2 - 0| = \left| 1 + (-1)^{2n} - (1 + (-1)^{2n+1}) \right|$$

$$\leq |1 + (-1)^{2n} - s| + |s - (1 + (-1)^{2n+1})|$$

$< 1 + 1 = 2$, a contradiction.

4)

Ross 8.5a

Let $\varepsilon > 0$. There exists N_1 such that $n > N_1$ implies $|a_n - s| < \varepsilon$

N_2 such that $n > N_2$ implies $|b_n - s| < \varepsilon$

Let $N = \max\{N_1, N_2\}$. If $n > N$,

$s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon$

so $|s_n - s| < \varepsilon$. Thus $s_n \to s$.

Ross 8.6a

For $n \in \mathbb{N}$, $-|s_n| \leq s_n \leq |s_n|$.

If $|s_n| \to 0$, $-|s_n| \to 0$, and so $s_n \to 0$ by 8.5a.

If $s_n \to 0$, Let $\varepsilon > 0$.

There exists $N \in \mathbb{R}$ such that $n > N$ implies $|s_n - 0| < \varepsilon$, so $||s_n| - 0| = |s_n - 0| < \varepsilon$. So $|s_n| \to 0$.

a) Ross 8.9

a) If $\lim s_n < a$ there exists $N \in \mathbb{R}$ such that $n > N$ implies

$$|s_n - \lim s_n| < a - \lim s_n$$

so $s_n < a$ ~~for~~ all $n > N$, which is infinitely many n.

b) If $\lim s_n > b$ there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$|s_n - \lim s_n| < \lim s_n - b. \text{ So}$$

$s_n > \lim s_n - (\lim s_n - b) = b$ for all $n > N$.

c) $\lim s_n \geq a$ and $\lim s_n \leq b$

so $\lim s_n \in [a, b]$.

b) <u>Ross 8.10</u>

Let $s = \lim s_n$, and chose $\varepsilon = s - a > 0$.
There exists $N \in \mathbb{R}$ such that
$n > N$ implies $|s_n - s| < \varepsilon$,
which in torn implies

$$s_n > s - \varepsilon = s - (s - a) = a.$$

b) For each $n \in \mathbb{N}$, choose $s_n \in \mathbb{Q}$ such that $r - \frac{1}{n} < s_n < r$.

Such an s_n exists because $\mathbb{Q}$ is dense in $\mathbb{R}$. Then (s_n) is a sequence and by squeeze thm since $(r - \frac{1}{n})$ and (r) converge to r, $s_n \to r$.

7) Ross 9.1

a) $\frac{n+1}{n} = 1 + \frac{1}{n}$ $\quad \lim (1+\frac{1}{n}) = 1 + \lim \frac{1}{n} = 1.$

b) $\frac{3n+7}{6n-5} = \frac{3+\frac{7}{n}}{6-\frac{5}{n}}$

$\lim (3+\frac{7}{n}) = 3 + 7\lim \frac{1}{n} = 3$

$6 - \frac{5}{n} > 0$ and $\lim (6-\frac{5}{n}) = 6 - 5\lim\frac{1}{n} = 6$

So $\lim \frac{3+\frac{7}{n}}{6-\frac{5}{n}} = \frac{3}{6} = \frac{1}{2}.$

c) $\frac{17n^5 + 73n^4 - 18n^2 + 3}{23n^5 + 13n^3}$

$= \frac{17 + 73(\frac{1}{n}) - 18(\frac{1}{n^3}) + 3(\frac{1}{n^5})}{23 + 13(\frac{1}{n^2})}$

$\lim (17 + 73(\frac{1}{n}) - 18(\frac{1}{n^3}) + 3(\frac{1}{n^5})) = 17$

$23 + 13(\frac{1}{n^2}) > 0$

$\lim (23 + 13(\frac{1}{n^2})) = 23$

So the limit is $\frac{17}{23}.$

Ross 9.2

a) $\lim(x_n+y_n) = \lim(x_n) + \lim(y_n) = 3+7=10$

b) Since $y_n^2 \neq 0$, $\lim y_n^2 = (\lim y_n)^2 = 49 \neq 0$

$\lim(3y_n - x_n) = 3\lim y_n - \lim x_n = 21-3=18$

$$\lim \frac{3y_n - x_n}{y_n^2} = \frac{18}{49}$$

8)

a) $S_{n+1}^2 - 2 = \frac{S_n^2}{4} + 1 + \frac{1}{S_n^2} - 2 = \frac{S_n^2}{4} - 1 + \frac{1}{S_n^2}$

$= \left(\frac{S_n}{2} - \frac{1}{S_n}\right)^2 \geq 0.$

b) $S_{n+1} - S_n = \frac{S_n}{2} + \frac{1}{S_n} - S_n = \frac{1}{S_n} - \frac{S_n}{2}$

$= \frac{2 - S_n^2}{2S_n} \leq 0$

since $S_n > 0$

(Indeed, $S_1 > 0$ and if $S_n > 0$, $\frac{S_n}{2} + \frac{1}{S_n} > 0$).

So S_n is decreasing, and thus bounded above by $S_1 = 2$ and below by 0. So (S_n) converges

c) The sequence (S_{n+1}) and (S_n) have the same limit (only the indices are different); call it S.

Since both converge and

$S_{n+1} = \frac{S_n}{2} + \frac{1}{S_n}$

Then $(s_{n+1})(s_n) = \frac{s_n^2}{2} + 1$

$$(\lim s_{n+1})(\lim s_n) = \frac{(\lim s_n)^2}{2} + 1$$

$$s^2 = \frac{s^2}{2} + 1$$

$s^2 = 2$,

So $s = +$ or $- \sqrt{2}$; since $s_n \geq 0$,

$s \geq 0$ and thus $s = \sqrt{2}$.