Homework 3

Due Monday, September 19 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. a) State what it means for a sequence (s_n) to converge to $s \in \mathbb{R}$.
   b) State what it means for a sequence (s_n) to not converge to $s \in \mathbb{R}$.
   c) Give an example of a sequence (s_n) such that $s_n < 0$ for all $n \in \mathbb{N}$ and $s_n \to 0$.
   d) Give an example of a bounded sequence which does not converge.

2. Ross 8.8

3. Prove that each of the following sequences does not converge to any $s \in \mathbb{R}$.
   a) (n)
   b) $\left(\cos\left(\frac{n\pi}{3}\right)\right)$
   c) $\left(\sin\left(\frac{n\pi}{3}\right)\right)$
   d) $(1 + (-1)^n)$

4. a) Ross 8.5, part a.
   b) Ross 8.6, part a.

5. a) Ross 8.9
   b) Ross 8.10

6. Let $r \in \mathbb{R}$. Prove that there exists a sequence (s_n) such that $s_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $s_n \to r$.

7. a) Ross 9.1
   b) Ross 9.2

8. Let s_n be defined inductively by $s_1 = 2$ and $s_{n+1} = \frac{s_n}{2} + \frac{1}{s_n}$.
   a) Show that $s_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$.
   b) Prove that s_n is monotone. Prove that s_n converges.
   c) Find $\lim s_n$ and justify your answer. Hint: Consider each side of the equation above as a sequence, and find its limit using limit theorems.

Extra (not graded): Assume, for this problem, that the completeness axiom fails (that is, there exists a nonempty set $S \subset \mathbb{R}$ with an upper bound but no supremum.) Shows that in that case there exists a sequence (s_n) of real numbers which is monotone and bounded but does not converge.