1) The supremum is the minimum of the set of upper bounds

The infimum is the maximum of the set of lower bounds.

a) $\sup [0,1] = 1 \in [0,1]$

b) $\sup (0,1) = 1 \notin (0,1)$

c) $\mathbb{N}$ has no upper bound.

(note! many other examples as well)

2)

Ross, 4.11.

Assume towards a contradiction that there exist $a, b \in \mathbb{R}$, $a < b$, such that $(a, b) \cap \mathbb{Q}$ is finite. Then $(a, b) \cap \mathbb{Q}$ has a maximum, which we call $r$. Since $r \in (a, b)$, $a < r < b$, and by the denseness of $\mathbb{Q}$ there exits $s \in \mathbb{Q}$ such that $r < s < b$. But then $s \in (a, b) \cap \mathbb{Q}$, which contradicts the definition of $r = \max(a, b) \cap \mathbb{Q}$.

3)

If $a > b$, then by the AP there exists $n \in \mathbb{N}$ such that $n > \frac{1}{a-b}$. Thus $a - b > \frac{1}{n}$ and $a > b + \frac{1}{n}$

4) We use the following fact:

If $a, b > 0$ then $a < b$ if and only if $a^2 < b^2$.

Proof: assume $a < b$. By axioms, $a^2 < ab$ and $ab < b^2$ so $a^2 < b^2$.

Similarly $a \geq b$ implies $a^2 \geq b^2$, so $a^2 < b^2$ must imply $a < b$.

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If $r \in S$, then $r^2 = |r|^2 < 2 < 2^2$ so $r \leq |r| < 2$. Thus $S$ is bounded above, and nonempty $(0, 1 \in S)$ so $t = \sup S$ exists. Since $1 \in S$, $t \geq 1 > 0$.

To show $t^2 = 2$, we rule out $t^2 > 2$ and $t^2 < 2$.

If $t^2 > 2$, let $0 < \delta < \min\left\{\frac{t^2-2}{2t}, 1\right\}$.

Then $(t-\delta)^2 = t^2 - 2t\delta + \delta^2 \geq t^2 - 2t\delta$

$> t^2 - (t^2 - 2) = 2$.

If $t^2 < 2$ let $0 < \delta < \min\left\{\frac{2-t^2}{2t+1}, 1\right\}$.

Then $(t+\delta)^2 = t^2 + 2t\delta + \delta^2 \leq t^2 + 2t\delta + \delta$ (since $\delta \leq 1$)

$= s^2 + \delta(2t+1) < t^2 + (2-t^2) = 2$

so $t + \delta \in S$, which contradicts $t = \sup S$.

5) (Note: other solutions, with different values, exist)

a) $\boxed{N_1 = 1}$ $n > 1 \Rightarrow 6n > 6 \Rightarrow 7n > 6 + n$

(if we assume $n \in \mathbb{N}$, any $N_1$ works)

b) $\boxed{N_2 = \sqrt{12}}$ $n > \sqrt{12} \Rightarrow n^2 > 12 \Rightarrow \frac{n^2}{2} > 6$

$\Rightarrow n^2 > 6 + \frac{n^2}{2} \Rightarrow n^2 - 6 > \frac{n^2}{2}$

(if we assume $n \in \mathbb{N}$, $N_2 = 3$ works)

c) $\boxed{N_3 = \sqrt{12}}$ $n > N_3 \Rightarrow n > N_1, N_2$ and $n^2 - 6 > 0$

$$\left| \frac{n+6}{n^2-6} \right| = \frac{n+6}{n^2-6} \leq \frac{7n}{\frac{n^2}{2}} = \frac{14}{n}.$$

(if we assume $n \in \mathbb{N}$, $N_3 = 3$ works)

d) $\boxed{N_4 = \frac{14}{\varepsilon}}$ If $n > \frac{14}{\varepsilon}$, $\frac{14}{n} < \varepsilon$.

e) Let $\varepsilon > 0$. Choose $N = \max\{\frac{14}{\varepsilon}, \sqrt{12}\}$.

If $n > N$, by a–d,

$\left| \frac{n+6}{n^2-6} - 0 \right| \leq \frac{14}{n} < \varepsilon$. So $\frac{n+6}{n^2-6} \to 0$.

6) a) $\boxed{N_1 = 1}$

$n > 1 \Rightarrow 6n > 6 \Rightarrow 7n > 6 + n$

b) $\boxed{N_2 = 6}$

$n > 6 \Rightarrow n - 6 > 0 \Rightarrow n^2 - 6n > 0$

$\Downarrow$

$n > 1 \Rightarrow 6n > 6 \Rightarrow n^2 - 6n \leq n^2 - 6$

c) $\boxed{N_3 = 6}$ if $n > 6$,

$n^2 - 6 > 0$, so

$$\left|\frac{n+6}{n^2-6}\right| = \frac{n+6}{n^2-6} < \frac{7n}{n^2-6n} = \frac{7}{n-6}$$

d) $N_{\cancel{3}} = \frac{7}{\varepsilon} + 6$

$n > \frac{7}{\varepsilon} + 6 \Rightarrow n - 6 > \frac{7}{\varepsilon}$

$\Rightarrow \varepsilon(n-6) > 7$

$\Rightarrow \varepsilon > \frac{7}{n-6}$

(since $n > \frac{7}{\varepsilon} + 6 > 6$).

e) Let $\varepsilon > 0$. Chose $N = \frac{7}{\varepsilon} + 6$.

For $n > N$, then

$\left| \frac{n+6}{n^2-6} - 0 \right| = \frac{n+6}{n^2-6} < \frac{7}{n-6} < \varepsilon$.

So $\frac{n+6}{n^2-6} \to 0$.

7)

# Ross 8.1

a) Let $\varepsilon > 0$. Choose $N = \frac{1}{\varepsilon}$. If $n > N$, then

$$\left| \frac{(-1)^n}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

b) Let $\varepsilon > 0$. Choose $N = \frac{1}{\varepsilon^3}$. If $n > N$, then

$$\left| \frac{1}{n^{1/3}} - 0 \right| = \frac{1}{n^{1/3}} < \frac{1}{N^{1/3}} = \varepsilon.$$

If $n^{1/3} \leq N^{1/3}$, then $(n^{1/3})^3 \leq (N^{1/3})^3$ so $n \leq N$.
Thus $n > N$ implies $n^{1/3} > N^{1/3}$.

c) Let $\varepsilon > 0$. Choose $N = \frac{7}{9\varepsilon}$. If $n > N$,

$$\left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| = \left| \frac{6n-3-(6n+4)}{9n+6} \right| = \left| \frac{-7}{9n+6} \right| < \frac{7}{9n}$$

$$< \frac{7}{9N} = \varepsilon.$$

d) See problem 2.

8)

## Ross 8.2

a) Let $\varepsilon > 0$. Choose $N = \frac{1}{\varepsilon}$. If $n > N$,

$$\left| \frac{n}{n^2+1} - 0 \right| < \frac{n}{n^2} = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

Thus $\frac{n}{n^2+1} \to 0$.

b) Let $\varepsilon > 0$. Choose $N = \frac{106}{9\varepsilon}$. If $n > N$

$$\left| \frac{7n-19}{3n+7} - \frac{7}{3} \right| = \frac{106}{9n+21} < \frac{106}{9n} < \frac{106}{9N} = \varepsilon$$

Thus $\frac{7n-19}{3n+7} \to \frac{7}{3}$

c) Let $\varepsilon > 0$. Choose $N = \frac{41}{14\varepsilon}$ If $n > N$

$$\left| \frac{4n+3}{7n-5} - \frac{4}{7} \right| = \frac{41}{49n-35} < \frac{41}{49n-35n} = \frac{41}{14n} < \frac{41}{14N} = \varepsilon$$

Thus $\frac{4n+3}{7n-5} \to \frac{4}{7}$

d) Let $\varepsilon > 0$. Choose $N = \frac{16}{25\varepsilon}$. If $n > N$

$$\left| \frac{2n+4}{5n+2} - \frac{2}{5} \right| = \frac{16}{25n+10} < \frac{16}{25n} < \frac{16}{25N} = \varepsilon$$

Thus $\frac{2n+4}{5n+2} \to \frac{2}{5}$

e) Let $\varepsilon > 0$. Choose $N = \frac{1}{\varepsilon}$. If $n > N$,

$$\left| \frac{1}{n} \sin n - 0 \right| = \frac{|\sin n|}{n} \le \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

Thus $\frac{1}{n} \sin n \to 0$.