1) Ross 3.3

<u>Theorem 3.1</u>

iv) By iii, and commutativity,

$(-a)(-b) = -(a(-b)) = -(-(ab))$

$ab + (-ab) = 0 = -(-ab) + (-(ab))$

So by i, $ab = -(-ab)$.

v) Since $c \neq 0$, c^{-1} exists.

$(ac)c^{-1} = (bc)c^{-1}$ so

$a(cc^{-1}) = b(cc^{-1})$ by associativity.

$a(1) = b(1)$ so

$a = b$.

Theorem 3.2

(v) $1 = 1^2$, so $0 \le 1$ by iv.

If $0 = 1$, then $1 + 0 = 1 + 1$, so $1 = 1 + 1$. But we have an axiom that $1 \neq n+1$ for any $n \in \mathbb{N}$, and $1 \in \mathbb{N}$. So $0 \neq 1$, and $0 < 1$.

vi) Since $a > 0$, $a^{-1} > 0$, and so $a < b$ implies $a a^{-1} < b a^{-1}$, or $1 < b a^{-1}$. Similarly, $0 < b^{-1}$, and $b^{-1} < b^{-1} b a^{-1} = a^{-1}$.

2) Ross 3.5 a)

$|b| \le a$ implies $-a \le b \le a$

Assume $|b| \le a$.

Note $0 \le |b| \le a$, and so $-a \le 0$.

Either $b \ge 0$ or $b < 0$.

If $b \ge 0$, then $-a \le 0 \le b = |b| \le a$.

If $b < 0$ $-a \le -|b| = b \le 0 \le a$.

Either way, $-a \le b \le a$.

$-a \le b \le a$ implies $|b| \le a$

Assume $-a \le b \le a$.

If $b \ge 0$, $|b| = b \le a$.

If $b < 0$, $|b| = -b \le -(-a) = a$.

Either way, $|b| \le a$.

Ross 3.5 b)

$|a| = |a+b-b| \leq |a+b| + |b|$

so $|a| - |b| \leq |a+b|$

$|b| = |b+a-a| \leq |b+a| + |a|$

so $|b| - |a| \leq |a+b|$

Since $||a|-|b|| = |a|-|b|$ or $|b|-|a|$,

$||a|-|b|| \leq |a+b|$.

Ross 4.13

(ii) is the definition of (iii), so
(ii) and (iii) are equivalent.
We show that (i) $\Leftrightarrow$ (ii).

$|a-b| < c \Rightarrow b-c < a < b+c$

Assume $|a-b| < c$. Using Ross 3.5a, we know
that $-c \le a-b \le c$

So $b-c \le a \le b+c$.
If $a = b+c$ then $|a-b| = |c| = c$
If $a = b-c$ then $|a-b| = |-c| = c$
Since those equalities are false,
$b-c < a < b+c$
(here we use $c > |a-b| \ge 0$)

$b-c < a < b+c \Rightarrow |a-b| < c$

Assume $b-c < a < b+c$.
Then $-c < a-b < c$, and by Ross 3.5a)
$|a-b| \le c$

If $|a-b| = c$ then either $a-b = c$

or $-(a-b) = c$ so $a-b = -c$.

Neither equality is true, so $|a-b| < c$.

3) a) If $n=1$, $|a_1|=|a_1|$ is immediate.

Now assume $|a_1+\dots+a_n|\leq|a_1|+\dots+|a_n|$ for some $n\in\mathbb{N}$. Then by the triangle inequality

$$|(a_1+\dots+a_n)+a_{n+1}|\leq|a_1+\dots+a_n|+|a_{n+1}|$$

and using our assumption we get

$$|a_1+\dots+a_n+a_{n+1}|\leq(|a_1|+\dots+|a_n|)+|a_{n+1}|.$$

b) By the triangle inequality

$$|a_1|=|(a_1+\dots+a_n)-(a_2+\dots+a_n)|$$
$$\leq|a_1+\dots+a_n|+|a_2+\dots+a_n|$$

and by part a)

$|a_2+\dots+a_n|\leq|a_2|+\dots+|a_n|$ so

$$|a_1+\dots+a_n|\geq|a_1|-|a_2|-\dots-|a_n|.$$

4) Ross 3.8

If $b < a$ then $2b < a+b < 2a$

so $b < \frac{a+b}{2} < a$.

Thus setting $b_1 = \frac{a+b}{2}$, we see

that $b_1 > b$ but $b_1 < a$.

So if $b_1 \geq a$ for all $b_1 > b$ it must be

the case that $b \geq a$.

5) Ross 4.7

a) $\inf S$, $\inf T$, $\sup S$, $\sup T$ all exist by the completeness axiom.

If $r \in S$, then $r \in T$, so $\inf T \leq r$. Thus $\inf T$ is a lower bound for S, and $\inf T \leq \inf S$.

Since S is nonempty, there exists $r \in S$, and $\inf S \leq r \leq \sup S$, so $\inf S \leq \sup S$.

If $r \in S$, $r \in T$, so $r \leq \sup T$. Thus $\sup T$ is an upper bound for S and $\sup S \leq \sup T$.

b) If $r \in S \cup T$, either $r \in S$, so
$r \leq \sup S \leq \max\{\sup S, \sup T\}$, or
$r \in T$, so $r \leq \sup T \leq \max\{\sup S, \sup T\}$.
So $\max\{\sup S, \sup T\}$ is an upper bound for $S \cup T$.
If M is an upper bound for $S \cup T$, it is an upper bound for S, and thus $M \geq \sup S$.

M is also an upper bound for

T, so $M \geq \sup T$. Thus

Thus $M \geq \max\{\sup S, \sup T\}$. This

proves $\max\{\sup S, \sup T\} = \sup(S \cup T)$.

b) <u>Ross 4.8</u>

a) Any element of T is an upper bound for S. Since T is nonempty, it has elements, and S is bounded above.

Any element of S is a lower bound for T. Since S is nonempty, it has elements, and T is bounded below.

b) Since S is nonempty and bounded above, it has a supremum.

Since T is nonempty and bounded below, it has an infimum.

If $t \in T$, t is an upper bound for S, so $\sup S \leq t$. Thus $\sup S$ is a lower bound for T, and $\sup S \leq \inf T$.

c) $S = [0,1]$ and $T = [1,2]$,

d) $S = (0,1)$ and $T = (1,2)$.

7) By definition, a is an upper bound for $\{r \in \mathbb{Q} \mid r < a\}$.

If $b < a$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r_0 \in \mathbb{Q}$ such that $b < r_0 < a$.

Then $r_0 \in \{r \in \mathbb{Q} \mid r < a\}$, so b is not an upper bound for $\{r \in \mathbb{Q} \mid r > a\}$.

Thus $a = \sup\{r \in \mathbb{Q} \mid r < a\}$.