1) Let $f_n, f : S \to \mathbb{R}$ be functions, for all $n \in \mathbb{N}$. Prove that $f_n \to f$ uniformly if and only if

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \mid x \in S\} = 0.$$

Suppose $f_n \to f$ uniformly. Let $\epsilon > 0$. Chose $N \in \mathbb{R}$ such that $n > N$, $x \in S$ implies $|f_n(x) - f(x)| < \epsilon/2$. Then

$$\sup\{|f_n(x) - f(x)| \ |x \in S\} \leq \epsilon/2 < \epsilon \text{ for all } n > N.$$

Thus $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \ |x \in S\} = 0$.

Now assume $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \ |x \in S\} = 0$. Let $\epsilon > 0$. Choose $N \in \mathbb{R}$ such that $n > N$ implies

$$\sup\{|f_n(x) - f(x)| \ |x \in S\} < \epsilon.$$

For $n > N, x \in S$

$$|f_n(x) - f(x)| \leq \sup\{|f_n(x) - f(x)| \ |x \in S\} < \epsilon$$

2)

For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \left(x - \frac{1}{n}\right)^2$. Find $f : [0,1] \to \mathbb{R}$ such that $f_n \to f$ uniformly, and prove your assertion.

Let $f(x) = x^2$.

$$|(x-1/n)^2 - x^2| = \left|-\frac{2x}{n} + \frac{1}{n^2}\right| \le \frac{2}{n} + \frac{1}{n^2} \le \frac{3}{n}$$

Thus

$$0 \le \sup\{|f_n(x) - f(x)| \; |x \in [0,1]\} \le \frac{3}{n}$$

and

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \; |x \in [0,1]\} = 0$$

by the squeeze theorem. Thus $f_n \to f$ uniformly.

3) For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = nx^n(1-x)$. Let $f : [0,1] \to \mathbb{R}$ be $f(x) = 0$. Prove that $f_n \to f$ pointwise, but not uniformly. Recall $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

If $x = 0, 1$, $f_n(x) = 0$ for all n, so $f_n(x) \to 0$.
If $x \in (0,1)$, $f_n(x) = nx^n(1-x) \to 0$ because $nx^n \to 0$: using a previous HW problem,

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = x \lim_{n\to\infty} \frac{n+1}{n} = x < 1$$

(or, if you want to use L'hospitals rule, which we did not cover and you are not expected to use Define $g : (0,\infty) \to \mathbb{R}$, $g(y) = yx^y$ (x is treated as a constant). By L'hopsital's rule, since x^{-y} and $(x^{-y})'$ are nonzero

$$\lim_{y\to\infty} g(y) = \lim_{y\to\infty} \frac{y}{x^{-y}} = \lim_{y\to\infty} \frac{1}{-\log(x)x^{-y}} = 0.$$

Thus $\lim_{n\to\infty} nx^n = \lim_{n\to\infty} g(n) = 0.$)

By solving $f'(0)$ se see that $f_n(x)$ achieves its maximum at $x = \frac{1}{1+\frac{1}{n}}$ so

$$\sup\{|f_n(x) - f(x)||x \in [0,1]\} = f_n\left(\frac{1}{1+\frac{1}{n}}\right) = n\left(\frac{1}{1+\frac{1}{n}}\right)^n \left(\frac{\frac{1}{n}}{1+\frac{1}{n}}\right) \to \frac{1}{e}$$

so f_n does not converge to f uniformly.

4)

Define $g_k : (0,1) \to \mathbb{R}$ by $g_k(x) = x^k$. Prove that $\sum_{k=0}^{\infty} g_k$ converges pointwise to $f(x) = \frac{1}{1-x}$ (that is, the sequence of partial sums converges pointwise). Prove that $\sum_{k=0}^{\infty} g_k$ does not converge uniformly.

The partial sums are given by

$$f_n(x) = \sum_{k=0}^{n} g_k = \frac{1-x^{n+1}}{1-x}.$$

These converge to $\frac{1}{1-x}$ for each $x \in (0,1)$ (for each x this is just a geometric series). However, for all $x \in (0,1)$

$$|f_n(x) - f(x)| = \frac{x^{n+1}}{1-x} \geq x^{n+1},$$

so for all n

$$\sup\{|f_n(x) - f(x)||x \in (0,1)\} \geq \left(\frac{1}{2^{\frac{1}{n+1}}}\right)^{n+1} = \frac{1}{2}$$

Thus f_n does not converge to f uniformly, and $\sum g_k$ does not converge uniformly.

5) Ross 25.3

$$\left|\frac{1}{2} - \frac{n + \cos x}{2n + \sin^2 x}\right| = \left|\frac{\sin^2 x - 2\cos x}{4n + 2\sin^2 x}\right| \leq \frac{3}{4n - 2} \to 0$$

so

$$\sup\left\{\left|\frac{1}{2} - f_n(x)\right| |x \in \mathbb{R}\right\} \to 0$$

and $f_n \to \frac{1}{2}$ uniformly. Thus

$$\lim_{n\to\infty} \int_2^7 f_n = \int_2^7 \frac{1}{2} = \frac{5}{2}.$$

6) For each $n \in \mathbb{N}$, define $f_n : (-1,1) \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.

a) Prove that (f_n) converges uniformly to $f(x) = |x|$.

b) Prove that f_n is differentiable and find f_n'.

c) Find the function $g : (-1,1) \to \mathbb{R}$ such that (f_n') converges pointwise to g. Prove that (f_n') does not converge uniformly to g.

a)
$$\left|\sqrt{x^2+\frac{1}{n}} - |x|\right| = \left|\frac{(\sqrt{x^2+\frac{1}{n}} - |x|)(\sqrt{x^2+\frac{1}{n}} + |x|)}{\sqrt{x^2+\frac{1}{n}} + |x|}\right| \le \left|\frac{x^2+\frac{1}{n} - |x|^2}{\frac{1}{\sqrt{n}}}\right| = \frac{1}{\sqrt{n}}$$

Thus
$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)||x \in (-1,1)\} \le \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$
and $f_n \to f$ uniformly.

b) $\sqrt{y}$ is differentiable when $y \neq 0$ and $x^2 + \frac{1}{n}$ is differentiable and nonzero so f_n is differentiable and
$$f_n'(x) = \frac{x}{\sqrt{x^2+\frac{1}{n}}}$$
by the chain rule.

c) Let $g(x) = 1$ for $x > 1$, $g(0) = 0$, and $g(x) = -1$ for $x < 1$.
For $x > 1$, $\lim_{n\to\infty} \frac{x}{\sqrt{x^2+\frac{1}{n}}} = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|} = 1$.
For $x = 0$, $f_n'(0) = 0$ for all n.
For $x < 0$, $\lim_{n\to\infty} \frac{x}{\sqrt{x^2+\frac{1}{n}}} = \frac{x}{|x|} = -1$.
So $f_n' \to g$ pointwise. However, g is not continuous, and each f_n' is continuous (the ratio of nonzero continuous functions), so the convergence cannot be uniform.

7) . Prove that $\sum_{n=1}^{\infty} nx^n$ converges to $\frac{x}{(1-x)^2}$ for $x \in (-1,1)$. Hint: (Use the fact that $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$).

The radius of convergence of $\sum_{k=0}^{\infty} x^n$ is 1, and for $x \in (-1,1)$

$$\sum_{k=0}^{\infty} x^n = f(x) = \frac{1}{1-x}.$$

By the theorem on derivatives of power series, for $x \in (-1,1)$

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} nx^{n-1}.$$

Fixing $x \in (-1,1)$, we cna multiply every elemet of the series by x, and the series will converge to x times the previous limit. So

$$\sum_{k=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

8)

a) Use the fact that for each $x \in \mathbb{R}$, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ to prove that $(e^x)' = e^x$.

Since the power series converges to e^x for all $x \in \mathbb{R}$, the radius of convergence must be $\mathbb{R}$. Thus the derivative of e^x is given by the power series formed by differentitation each element in the original power series:

$$(e^x)' = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x.$$

b) Define $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Find a power series which converges at each $x \in \mathbb{R}$ to $\int_0^x f$.

Fix $x \in \mathbb{R}$. Let $y = -x^2$. We know that

$$e^{-x^2} = e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

This is a power series in x with coefficients $a_{2n} = (-1)^n/n!$ and $a_{2n+1} = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since the equation is true for any $x \in \mathbb{R}$, the radius of convergence of this power series must be ∞. Thus we can integrate term by term, and

$$\int_0^x f = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}.$$

This in turn is a power series with coeeficients $a_{2n+1} = \frac{(-1)^n}{(2n+1)n!}$ and $a_{2n} = 0$ for all $n \in \mathbb{N} \cup \{0\}$.

9) Prove that there does not exists a power series which converges pointwise to $f : (-1, 1) \to \mathbb{R}$, $f(x) = |x|$

If such a power series existed, it would have positive radius of convergence, and thus the limit would be differentiable at 0. But $|x|$ is not differentiable at 0. (This follows from using the definition of the derivative, from Ross 29.17, or from the following argument: If $|x|$ were differentiable at 0, the image of the derivative would be an interval by the intermediate value theorem for derivatives, but it would also be $\{-1, 1, f'(0)\}$, which cannot be an interval).