Homework 12

Due Friday, December 9 at 10am. Please upload a legible copy to Gradescope.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Let $f_n, f : S \to \mathbb{R}$ be functions, for all $n \in \mathbb{N}$. Prove that $f_n \to f$ uniformly if and only if

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \mid x \in S\} = 0.$$

2. For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \left(x - \frac{1}{n}\right)^2$. Find $f : [0,1] \to \mathbb{R}$ such that $f_n \to f$ uniformly, and prove your assertion.

3. For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = nx^n(1-x)$. Let $f : [0,1] \to \mathbb{R}$ be $f(x) = 0$. Prove that $f_n \to f$ pointwise, but not uniformly. Recall $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.

4. Define $g_k : (0,1) \to \mathbb{R}$ by $g_k(x) = x^k$. Prove that $\sum_{k=0}^{\infty} g_k$ converges pointwise to $f(x) = \frac{1}{1-x}$ (that is, the sequence of partial sums converges pointwise). Prove that $\sum_{k=0}^{\infty} g_k$ does not converge uniformly.

5. Ross 25.3

6. For each $n \in \mathbb{N}$, define $f_n : (-1,1) \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.
   a) Prove that (f_n) converges uniformly to $f(x) = |x|$.
   b) Prove that f_n is differentiable and find f_n'.
   c) Find the function $g : (-1,1) \to \mathbb{R}$ such that (f_n') converges pointwise to g. Prove that (f_n') does not converge uniformly to g.

7. Prove that $\sum_{n=1}^{\infty} nx^n$ converges to $\frac{x}{(1-x)^2}$ for $x \in (-1,1)$. Hint: (Use the fact that $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$).

8. Recall that for $x \in \mathbb{R}$, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
   a) Prove that $(e^x)' = e^x$.
   b) Define $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Find a power series which converges at each $x \in \mathbb{R}$ to $\int_0^x f$.

9. Prove that there does not exists a power series which converges pointwise to $f : (-1,1) \to \mathbb{R}$, $f(x) = |x|$