1) Ross 32.8

Let $\varepsilon > 0$. There exists a partition P_ε of $[a,b]$ such that

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$$

Since this inequality remains true if we add points to P_ε, we may assume $c, d \in P_\varepsilon$ so

$$P_\varepsilon = \{a = t_0 < \dots < t_s = c < \dots < t_r = d < \dots < b = t_n\}$$

Then $Q_\varepsilon = \{t_s = c < \dots < t_r = d\} \subset P_\varepsilon$ is a partition of $[c,d]$ and

$U(f, Q_\varepsilon) - L(f, Q_\varepsilon)$

$$= \sum_{k=s+1}^{r} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1})$$

$$\leq \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1})$$

$= U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$. So $f:[c,d] \to \mathbb{R}$ is int.

2) Let P_n be the partition with $t_k = a + \left(\frac{b-a}{n}\right)k$, which has n intervals of length $\frac{b-a}{n}$.

Assume $n > K$. At most K of the intervals contain an element $s_i \in S$. On those intervals,

$0 \le M(f, [t_{k-1}, t_k]) \le M(f, [a,b])$

$0 \ge m(f, [t_{k-1}, t_k]) \ge m(f, [a,b])$

For the others, $f = 0$, so M and m are 0.

Thus

$$0 \le U(f, P_n) \le M(f, [a,b]) \left(\frac{b-a}{n}\right) K$$

$$0 \ge L(f, P_n) \ge m(f, [a,b]) \left(\frac{b-a}{n}\right) K$$

By the squeeze theorem

$U(f, P_n) \to 0$, $L(f, P_n) \to 0$ so

f is int. and $\int_a^b f = 0$.

3) 33.7 Let $S \subseteq [a, b]$

For all $x, y \in S$

$$f^2(x) - f^2(y) = (f(x)+f(y))(f(x)-f(y))$$
$$\leq (f(x)+f(y))(M(f,S) - m(f,S))$$
$$\leq \underbrace{2B(M(f,S) - m(f,S))}$$

define this to be C, which does not depend on x or y

Fix $y \in S$. Since

$f^2(x) \leq C + f^2(y)$ for all $x \in S$,

$M(f^2, S) \leq C + f^2(y)$.

Thus for all $y \in S$,

$f^2(y) \geq M(f^2, S) - C$, and so

$m(f^2, S) \geq M(f^2, S) - C$.

Thus

$$M(f^2, S) - m(f^2, S) \leq 2B\left(M(f,S) - m(f,S)\right).$$

Let $\varepsilon > 0$. There is a partition P_ε of $[a,b]$

Such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \frac{\varepsilon}{2B}.$$

Then

$$U(f^2, P_\varepsilon) - L(f^2, P_\varepsilon)$$

$$= \sum_{k=1}^{n} \left[M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \right] (t_k - t_{k-1})$$

$$\leq \sum_{k=1}^{n} 2B \left\{ M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \right] (t_k - t_{k-1})$$

$$= 2B \left(U(f, P_\varepsilon) - L(f, P_\varepsilon) \right) < \varepsilon.$$

So f^2 is integrable.

This is a) and b) together.

33.8 a) If f, g are integrable, then $f+g$, $f-g$ are integrable, so by above $(f+g)^2, (f-g)^2$ are integrable, and thus

$$\frac{1}{4}\left((f+g)^2 - (f-g)^2\right) = \frac{1}{4}\left(f^2 + 2fg + g^2 - f^2 + 2fg - g^2\right) = fg$$

is integrable.

4) We prove the contrapositive:

if $f(x) \geq 0$ for all x, $f(x_0) \neq 0$ for some $x_0 \in [a,b]$, and f is continuous, then

$$\int_a^b f \neq 0.$$

Assume $f(x_0) > 0$; the other case is similar.

Let $\varepsilon = \frac{f(x_0)}{2} > 0$. There exists $\delta > 0$ such that if $|x - x_0| < \delta$, $f(x) > f(x_0) - \varepsilon = \varepsilon$

(we can make δ smaller if necessary so $(x_0 - \delta, x_0 + \delta) \subset [a,b]$. Also easy to fix if $x_0 = a$ or b)

Then

$$\int_a^b f = \int_a^{x_0-\delta} f + \int_{x_0-\delta}^{x_0+\delta} f + \int_{x_0+\delta}^{b} f$$ since $f \geq 0$

$$\geq \int_{x_0-\delta}^{x_0+\delta} f \geq \int_{x_0-\delta}^{x_0+\delta} \varepsilon = 2\varepsilon\delta > 0.$$

5) Let $f(t) = e^{t^2}$, $F(x) = \int_0^x f$

$$\lim_{x \to 0} \frac{1}{x} \int_0^x f = \lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = F'(0) = f(0)$$
$$= 1$$

$$\lim_{h \to 0} \frac{1}{h} \int_3^{3+h} f = \lim_{h \to 0} \frac{F(3+h) - F(3)}{3 + h - 3}$$

$$= F'(3) = f(3) = e^9.$$

6) a) $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & x \in [0,1] \\ 4x - \frac{7}{2} & x > 1 \end{cases}$

b) F is continuous by FTC part $\underline{\text{II}}$, or by checking.

c) f is a polynomial except at 0 and 1. at 0, the limits on the left + right match, so f is continuous (HW problem).

Thus by FTC II, F is diff except maybe at 1, and

$$F' = \begin{cases} 0 & x < 0 \\ x & x \in [0,1) \\ 4 & x > 1 \end{cases}$$

By a HW problem, such a function is not diff at 1 because

$x' = 1 \neq 4' = 0$ at 1.

7) Let $G(x) = \int_0^x f$; G is diff since f is cont.

Then $F(x) = \int_0^{x+1} f + \int_{x-1}^0 f = G(x+1) - G(x-1)$

is diff and

$$F' = G'(x+1)(x+1)' - G'(x-1)(x-1)'$$
$$= f(x+1) - f(x-1)$$

8) define $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{|x|}$. Then f is continuous.

define $u: \mathbb{R} \to \mathbb{R}$, $u(x) = 1 - x^2$. Then u is diff and $u' = -2x$ is continuous.

So

$$\int_0^1 (f \circ u) u' = \int_0^1 \sqrt{|1 - x^2|}\,(-2x)$$

$$= -2 \int_0^1 x\sqrt{1-x^2} = \int_{u(0)}^{u(1)} f$$

$$= \int_1^0 \sqrt{|x|} = \int_1^0 \sqrt{x} = \frac{2}{3}\, 0^{\frac{3}{2}} - \frac{2}{3}\, 1^{\frac{3}{2}} = -\frac{2}{3}$$

Thus $\displaystyle\int_0^1 x\sqrt{1-x^2} = \frac{1}{3}$