1) If $x=y$, the inequality is true.

If $x<y$, since $\cos$ is differentiable on $\mathbb{R}$, there exists $c \in (x,y)$ such that

$$\frac{\cos(y)-\cos(x)}{y-x} = \cos'(c) = -\sin(c)$$

Thus since $\sin(c) \in [-1,1]$,

$$\left| \frac{\cos(x)-\cos(y)}{x-y} \right| = |\sin(c)| \leq 1.$$

2) Ross 29.5

Let $a \in \mathbb{R}$. We first prove that f is differentiable at a and $f'(a) = 0$. Let $\epsilon > 0$. Let $\delta = \epsilon$. If $x \in \mathbb{R} \backslash \{a\}$ and $|x - a| < \delta$ then

$$\left| \frac{f(x) - f(x)}{x - a} \right| \leq \frac{(x-a)^2}{|x-a|} = |x - a| < \delta = \epsilon.$$

Thus $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0$ and $f'(a) = 0$. Since this is true for all $a \in \mathbb{R}$, f is constant.

3) Ross 29.13

Let $h(x) = g(x) - f(x)$. By Derivative theorems, h is differentiable and $h'(x) = g'(x) - f'(x) \geq 0$. Thus h is nondecreasing, and in particular $h(x) \geq h(0) = 0$ for all $x \geq 0$. Thus $g(x) \geq f(x)$ for all $x \geq 0$.

4) Ross 29.17

Let $\epsilon > 0$. There exists δ_1 such that if $x \in I \backslash \{a\}$ and $|x-a| < \delta_1$ then

$$\left| \frac{g(x)-g(a)}{x-a} - g'(a) \right| < \epsilon$$

There exists δ_2 such that if $x \in I \backslash \{a\}$ and $|x-a| < \delta_1$ then

$$\left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < \epsilon$$

First, assume that $f(a) = g(a)$ and $f'(a) = g'(a)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Let $x \in I \backslash \{a\}$ such that $|x-a| < \delta$. If $x > a$ then since $|x-a| < \delta \leq \delta_1$

$$\left| \frac{h(x)-h(a)}{x-a} - g'(a) \right| = \left| \frac{g(x)-g(a)}{x-a} - g'(a) \right| < \epsilon.$$

If $x < a$ then since $|x-a| < \delta \leq \delta_2$ and $f(a) = g(a)$

$$\left| \frac{h(x)-h(a)}{x-a} - g'(a) \right| = \left| \frac{f(x)-g(a)}{x-a} - g'(a) \right| = \left| \frac{f(x)-g(a)}{x-a} - f'(a) \right| < \epsilon.$$

In either case,

$$\left| \frac{h(x)-h(a)}{x-a} - g'(a) \right| < \epsilon.$$

Thus $\lim_{x \to a} \frac{h(x)-h(a)}{x-a} = g'(a)$ and h is differentiable at a.

Now assume $f(a) \neq g(a)$. Let $x_n = a - \frac{1}{n}$ so $x_n \to a$ Since $x_n < a$, $h(x_n) = f(x_n)$. Since f is differentiable at a, f is continuous at a, and $f(x_n) \to f(a)$. But $f(a) \neq g(a) = h(a)$, so $h(x_n)$ does not converge to $h(a)$ and h is not continuous at a. A differentiable function is continuous so h is not differentiable at a.

Finally, assume $g(a) = f(a)$ but $f'(a) \neq g'(a)$. Let $x_n = a + \frac{(-1)^n}{n}$ for all $n \in \mathbb{N}$. Then for large n $x_n \in I \backslash \{a\}$ and $x_n \to a$. But $x_{2n} > a$ so

$$\frac{h(x_{2n})-h(a)}{x_{2n}-a} = \frac{g(x_{2n})-g(a)}{x_{2n}-a} \to g'(a)$$

while $x_{2n-1} < a$ so

$$\frac{h(x_{2n-1})-h(a)}{x_{2n-1}-a} = \frac{f(x_{2n-1})-g(a)}{x_{2n-1}-a} = \frac{f(x_{2n-1})-f(a)}{x_{2n-1}-a} \to f'(a).$$

Since all subsequences of a covnergent sequence converge to the same limit, $\frac{h(x_n)-h(a)}{x_n-a}$ does not converge. Thus h is not differentiable at a.

5) Suppose there exist $x, y \in I$ such that $f'(x) < 0$ and $f'(y) > 0$. Then by the intermediate value thm for derivatives there exists $z \in I$ such that $f'(z) = 0$, which contradicts the assumption. Thus either $f'(x) < 0$ for all $x \in I$, and f is strictly decreasing, or $f'(x) > 0$ for all $x \in I$ an f is strictly increasing.

6) Ross, 32.6

Let $U_n = U(f, P_n)$ and $L_n = L(f, Q_n)$ for partitions P_n, Q_n, and assume $\lim_{n\to\infty}(U_n - L_n) = 0$. Let $\epsilon > 0$. There exists N such that for $n > N$ $U_n - L_n < \epsilon$. then

$$U(f, P_n \cup Q_n) - L(f, P_n \cup Q_n) \leq U(f, P_n) - L(f, Q_n) < \epsilon.$$

So f is integrable. Since $U_n \geq \int_a^b f \geq L_n$, for $n > N$

$$\left| U_n - \int_a^b f \right| < \epsilon$$

and

$$\left| L_n - \int_a^b f \right| < \epsilon$$

so $\lim U_n = \lim L_n = \int_a^b f$.

7) Let $a, b \in \mathbb{R}$, $a < b$. Prove that $f : [a,b] \to \mathbb{R}$, $f(x) = x$ is integrable, and find $\int_a^b f$.

Let
$$P_n = \left\{ a, a + \frac{b-a}{n}, a + 2\frac{(b-a)}{n}, ..., a + (n-1)\frac{(b-a)}{n}, b \right\}.$$
In other words, $P_n = \{t_0 < t_1 < ... < t_n\}$ where $t_k = a + k\frac{b-a}{n}$. Then
$$U(f, P_n) = \sum_{k=1}^{n} \left(a + k\frac{b-a}{n} \right) \frac{b-a}{n} = a(b-a) + \frac{(b-a)^2}{n^2} \sum_{k=1}^{n} k = a(b-a) + (b-a)^2 \frac{n(n-1)}{2n^2}$$
$$L(f, P_n) = \sum_{k=1}^{n} \left(a + (k-1)\frac{b-a}{n} \right) \frac{b-a}{n} = a(b-a) - \frac{(b-a)^2}{n} + \frac{(b-a)^2}{n^2} \sum_{k=1}^{n} k$$
$$= a(b-a) - \frac{(b-a)^2}{n} + (b-a)^2 \frac{n(n-1)}{2n^2}.$$
The rest follows from problem 5. In detail:
Let $\epsilon > 0$. Chose $n > (b-a)^2/\epsilon$. Then
$$U(f, P_n) - L(f, P_n) = \frac{(b-a)^2}{n} < \epsilon.$$
So f is integrable. It follows that for $n > (b-a)^2/\epsilon$,
$$\left| U(f, P_n) - \int_a^b f \right| < \epsilon$$
so
$$\int_a^b f = \lim_{n \to \infty} U(f, P_n) = \frac{b^2}{2} - \frac{a^2}{2}.$$

8) By density of $\mathbb{R} \setminus \mathbb{Q}$, every interval contains an irrational number x at which $f(x)=0$, and $0 \le f(y)$ for all $y \in [0,1]$. So

$m(f, [t_{k-1}, t_k]) = 0$ for any t_{k-1}, t_k, and $L(f, P) = 0$ for any partition P of $[0,1]$, and thus $L(f) = 0$.

$f([t_{k-1}, t_k]) \subseteq [t_{k-1}, t_k] \cup \{0\}$, so t_k is an upper bound for $f([t_{k-1}, t_k])$. If $r < t_k$, there exists $x \in \mathbb{Q}$ such that $\max\{t_{k-1}, r\} < x < t_k$ (density of $\mathbb{Q}$) and thus $r < x = f(x) \in f([t_{k-1}, t_k])$, so r is not an upper bound for $f([t_{k-1}, t_k])$.
Thus $M(f, [t_{k-1}, t_k]) = t_k$.

let $: [0,1] \to \mathbb{R}$ be $g(x) = x$. Since g is increasing,

$M(g, [t_{k-1}, t_k]) = t_k = M(f, [t_{k-1}, t_k])$. Thus $U(f, P) = U(g, P)$ for all partitions P, and thus $U(f) = U(g) = \int_0^1 g = \frac{1}{2}$.

Since $U(f) = \frac{1}{2} \neq 0 = L(f)$, f is not integrable.