1) Define $F : [a,b] \to \mathbb{R}$

$$F(x) = \int_a^x f - g.$$

Then $F(a) = F(b) = 0$.

Since $f-g$ is continuous, by the Fund. Thm. part II

- $F$ is cont. on $[a,b]$
- $F$ is differentiable on $(a,b)$
- $F'(x) = f(x) - g(x)$ for $x \in (a,b)$.

By the MVT, there exists $x_0 \in (a,b)$ such that

$$F'(x_0) = f(x_0) - g(x_0) = \frac{F(b) - F(a)}{b-a} = 0.$$

Alternate Solution

If $f(x) \geq g(x)$ for all $x \in [0,1]$ or $g(x) \geq f(x)$ for all $x \in [0,1]$

then $\int_a^b f - g = \int_a^b g - f = 0$ implies $f(x) = g(x)$ for all $x \in [0,1]$

since $f-g$ is continuous.

If neither case holds, then there exist $x_1$ such that $f(x_1) > g(x_1)$ and $x_2$ such that $g(x_2) > f(x_2)$. Applying the intermediate value theorem to $f-g$ there exists $x_0$ between $x_1, x_2$ where $f(x_0) = g(x_0)$.

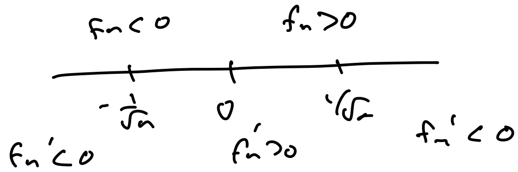
a)

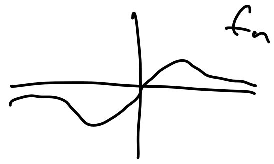
Let $f(x)=0$ for all $x\in\mathbb{R}$.

$$f_n'(x): \frac{1+nx^2 - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

$f_n'(x)=0$ exactly when $x=\pm\frac{1}{\sqrt{n}}$.

Analysis of the sign of $f_n$, $f_n'$ shows $f_n$ achieves its min at $-\frac{1}{\sqrt{2}}$, max at $\frac{1}{\sqrt{2}}$




So $\sup\{|f_n(x)| \mid x\in\mathbb{R}\} = |f_n(\pm\frac{1}{\sqrt{n}})| = \frac{1}{2\sqrt{n}} \to 0$ as $n\to\infty$. Thus $f_n\to f$ uniformly.

$f'=0$. If $f_n'(0)=1 \to 1 \neq f'(0)$ as $n\to\infty$ but

~~If~~ $x\neq 0$, $f_n'(x) = \frac{1/n^2 - x^2/n}{(\frac{1}{n}+x^2)^2} \to 0 = f'(0)$.

3) a) If $x \in \{\pi, -\pi, 0\}$, $\sin(x) = 0$,

so $\sum g_k(0) = 0$

otherwise, $|\cos(x)| < 1$, so the

geometric series $\sum_{k=0}^{\infty} (\cos^2(x))^k = \frac{1}{1-\cos^2(x)} = \frac{1}{\sin^2(x)}$

and $\sum_{k=0}^{\infty} g_k(x) = \sin^2(x) \sum_{k=0}^{\infty} (\cos^2(x))^k = 1$.

So $f(x) = \begin{cases} 0 & x \in \{\pi, -\pi, 0\} \\ 1 & x \notin \{\pi, -\pi, 0\} \end{cases}$.

b) Since each $g_k$ is continuous, if the convergence were uniform, $f$ would be continuous. So the convergence is not uniform.

c) For $x \in [\pi/4, \frac{3\pi}{4}]$, $|\cos(x)| \le \frac{1}{\sqrt{2}}$

So $|g_k(x)| \le \frac{1}{2^k}$. Since $\sum \frac{1}{2^k}$ converges, the M test implies $\sum g_k$ converges uniformly.

4) Define $F: \mathbb{R} \to \mathbb{R}$ by $F(y) = \int_0^y f$. Since $f$ is continuous,

$F$ is differentiable and $F' = f$ by the fundamental thm. Let $u(x) = x^2$.

Then $g = F \circ u$ so

$$g'(x) = F'(u(x))\, u'(x) = e^{\sin(x^2)}(2x).$$

5) Let $R$ be the radius of convergence of $\sum a_n x^n$. If $R < 2$, then $\sum a_n 2^n$ does not converge. So $R \geq 2$.

Since $1 < 2$, $\sum a_n x^n$ converges uniformly on $[-1, 1]$.

6) a) By the IVT for derivatives,

$f$ is either increasing or decreasing.

If $f$ is increasing, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Thus it is not possible that $f(x_1) = f(x_2)$

for $x_1 \neq x_2$. So for all $y \in f(I)$,

$f^{-1}(y)$ contains exactly one element $x \in I$;

define $g : f(I) \to \mathbb{R}$ by $g(y) = x$.

Then $g(f(x)) = g(y) = x$ since $\{x\} = f^{-1}(y)$

by definition. Similarly, $f(g(y)) = f(x) = y$.

b) first, not $g$ is increasing. If

$y_1 < y_2$, $g(y_1) \not< g(y_2)$, because

otherwise $g(y_1) \geq g(y_2)$ while

$f(g(y_1)) = y_1 < y_2 = f(g(y_2))$, which is

false since $f$ is increasing.

Now let $y_0 \in f(I)$ and $\varepsilon > 0$.

Let $x_0 = g(y_0)$. Since

$$x_0 - \varepsilon < x_0 < x_0 + \varepsilon,$$

$$f(x_0 - \varepsilon) < f(x_0) = y_0 < f(x_0 + \varepsilon)$$

so choosing $0 < \delta < \min\{y_0 - f(x_0 - \varepsilon), f(x_0 + \varepsilon) - y_0\}$

we have

$$f(x_0 - \varepsilon) < y_0 - \delta < y_0 < y_0 + \delta < f(x_0 + \varepsilon).$$

Then if $|y - y_0| < \delta$, $y_0 - \delta < y < y_0 + \delta$, so

since $g$ is increasing,

$$g(f(x_0 - \varepsilon)) = x_0 - \varepsilon < g(y) < x_0 + \varepsilon = g(f(x_0 + \varepsilon))$$

Thus $|g(y) - g(y_0)| = |g(y) - x_0| < \varepsilon$.