

20)

Choose $\varepsilon = 1$. Let $\delta > 0$. Choose $x = \frac{1}{\delta}$, $y = x + \frac{\delta}{2}$.

Then $|x - y| < \delta$ but

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| = \left(\frac{1}{\delta} + \frac{\delta}{2}\right) \frac{\delta}{2}$$

$$\geq \left(\frac{1}{\delta}\right) \left(\frac{\delta}{2}\right) = 1.$$

(idea: If $x, y > 0$, $|x - y| = \frac{\delta}{2}$, then

$$|x + y||x - y| = |x + y| \frac{\delta}{2} \leq x \frac{\delta}{2}; \text{ so need}$$

$$x \frac{\delta}{2} \geq 1. \text{ solve for } x.)$$

21) $f: [0, 2] \rightarrow \mathbb{R}$ is continuous.

Note $h: [0, 1] \rightarrow \mathbb{R}$, $h(x) = x + 1$ is continuous.

Since $h([0, 1]) = [1, 2] \subset [0, 2]$,

$f \circ h: [0, 1] \rightarrow \mathbb{R}$ is continuous. Also,

the restriction $f: [0, 1] \rightarrow \mathbb{R}$ is continuous.

$$\text{So } g: [0, 1] \rightarrow \mathbb{R}, \quad g(x) = f \circ h(x) - f(x) \\ = f(x+1) - f(x)$$

is continuous.

$$g(0) = f(1) - f(0) \quad \text{and} \quad g(1) = f(2) - f(1) \\ = f(0) - f(1).$$

It follows that either $= -g(0)$

a) $g(0) > 0$, $g(1) < 0$

b) $g(1) < 0$, $g(0) > 0$

c) $g(1) = g(0) = 0$.

In cases a and b, 0 is between $g(0)$

and $g(1)$. Since $[0, 1]$ is an interval, there

exists $x_0 \in (0, 1)$ such that $g(x) = 0$.

In case c) take $x_0 = 0$.

$$\text{Thus } f(x_{0+1}) - f(x_0) = 0$$

$$\text{So } f(x_{0+1}) = f(x_0)$$

$$\text{and } |(x_{0+1}) - x_0| = 1.$$

22)

a) $f(x) = 3x + 11$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. Let $|x-y| < \delta$.

Then $|f(x) - f(y)| = |3x + 11 - (3y + 11)| = 3|x-y| < 3\delta = \varepsilon$.

b) $f: [0, 3] \rightarrow \mathbb{R}$, $f(x) = x^2$

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{6}$.

Let $x, y \in [0, 3]$, $|x-y| < \delta$. Then

$$|x^2 - y^2| = |x+y||x-y| < 6\delta = \varepsilon.$$

c) $f: [\frac{1}{2}, \infty) \rightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{4}$. If $x, y \in [\frac{1}{2}, \infty)$,

$|x-y| < \delta$, then

$$|\frac{1}{x} - \frac{1}{y}| = \frac{|x-y|}{xy} \leq 4|x-y| < 4\delta = \varepsilon.$$

$$23) a) f: [0, 2] \rightarrow \mathbb{R}, \quad f(x) = \frac{x}{x+1}$$

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$

If $x, y \in [0, 2]$, $|x-y| < \delta$ then

$$\left| \frac{x}{x+1} - \frac{y}{y+1} \right| = \left| \frac{xy+x-xy-y}{(x+1)(y+1)} \right| = \frac{|x-y|}{(x+1)(y+1)} \leq |x-y| < \delta = \varepsilon.$$

$$b) f: [1, \infty) \rightarrow \mathbb{R}, \quad f(x) = \frac{5x}{2x-1}$$

Let $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{5}$

If $x, y \in [1, \infty)$ and $|x-y| < \delta$

$$\left| \frac{5x}{2x-1} - \frac{5y}{2y-1} \right| = \left| \frac{5y-5x}{(2x-1)(2y-1)} \right| \leq 5|x-y| < 5\delta = \varepsilon.$$

$$24) f: (0, 1) \rightarrow \mathbb{R}, \quad f(x) = x^2$$

Choose $\varepsilon = 1$, let $\delta > 0$,

If $\delta < 1$,

$$\text{Choose } x = (\delta)^{1/3} \quad y = x - \frac{\delta}{2}.$$

Then $0 < \delta < x < 1$ and

$$0 < \frac{\delta}{2} < y < x < 1, \text{ so } x, y \in (0, 1)$$

and $|x - y| = \frac{\delta}{2} < \delta$, but

$$\left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \frac{|x+y||x-y|}{x^2 y^2} \geq \frac{2y \left(\frac{\delta}{2}\right)}{x^2 y^2}$$

$$= \frac{\delta}{x^2 y} \geq \frac{\delta}{x^3} = 1 = \varepsilon.$$

If $\delta \geq 1$, choose $x = \frac{1}{2}$ and $y = \frac{1}{3}$

$$\text{so } |x-y| = \frac{1}{6} < \delta \text{ and } \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = 5 > \epsilon.$$