Math 104 002 **Midterm** **October 13, 2022**

Name: ______________________________

SID: ______________________________

Instructions :

1. You have 80 minutes, 7:10pm-8:30pm.
2. No books, notes, or other outside materials are allowed.
3. There are 6 questions on the exam.
4. You need to show all of your work and justify all statements. If you need more space, use the pages at the back of the exam or come get more paper at the front of the class. If you do so, please indicate which page your solution continues on.
5. Before you begin, take a quick look at all the questions on the exam, and start with the one you feel the most comfortable solving. It is more important to do the problems well that you know how to do, than it is to finish the whole exam.
6. While attempting any problem, do write something even if you are unable to solve it completely. You may get partial credit.

(Do not fill these in; they are for grading purposes only.)

1	
2	
3	
4	
5	
6	
Total	

1. (10 points) Find the limit s of the sequence $\left(\frac{3n^3+2}{2n^3-n}\right)$. Prove that the sequence converges to s using only the definition of convergence (no theorems or other limits).

$$\frac{3n^3+2}{2n^3-n} \longrightarrow \frac{3}{2}$$

Let $\varepsilon > 0$. Choose $N = \sqrt{\frac{7}{2\varepsilon}}$

If $n \in \mathbb{N}$, $n > N$ then

$$\left|\frac{3n^2+2}{2n^3-n} - \frac{3}{2}\right| = \left|\frac{4+3n}{4n^3-2n}\right| \leq \frac{7n}{4n^3-2n}$$

sine $n \geq 1$, $4n \geq 4$

$$= \frac{7}{4n^2-2} \leq \frac{7}{2n^2} < \frac{7}{2N^2} = \varepsilon.$$

sine $2 \leq 2n^2$

2. Let S and T be bounded subsets of $(0, \infty)$. Define

$$R = \{st \mid s \in S, t \in T\}$$

(a) (5 points) Prove that there exists a sequence in R which converges to $\sup(S)\sup(T)$.

(b) (5 points) Prove that $\sup(R) = \sup(S)\sup(T)$.

a) There exists a sequence (s_n) in S, $s_n \to \sup S$
There exists a sequence (t_n) in T, $t_n \to \sup T$
Thus $s_n t_n \in R$ and $s_n t_n \to \sup S \sup T$.

b) If $st \in R$, since $s, t > 0$,
$st \leq \sup(S)\, t \leq \sup(S) \sup(T)$.
So $\sup(S)\sup(T)$ is an UB for R
and $\sup(R) \leq \sup(S) \sup(T)$.
Let $r < \sup(S)\sup(T)$. Then there exists $N \in \mathbb{R}$
such that if $n > N$,
$|s_n t_n - \sup(S)\sup(T)| < \sup(S)\sup(T) - r$.
So $s_n t_n > r$, and thus r is not an UB
for R. Thus $\sup(R) \geq \sup(S)\sup(T)$

3. (5 points) Let $L = \limsup((s_n))$. Let $\epsilon > 0$. Prove that there exists $N \in \mathbb{R}$ such that $n \in \mathbb{N}$, $n > N$ implies $s_n < L + \epsilon$.

Let $V_N = \sup T_N = \sup\{s_n \mid n \in \mathbb{N}, n > N\}$.

Then $V_N \to L$, so there exists $N \in \mathbb{R}$

such that $|V_N - L| < \varepsilon$

which implies $V_N < L + \varepsilon$.

Thus if $n \in \mathbb{N}$, $n > N$, $s_n \leq V_N < L + \varepsilon$.

4. (10 points) Define

$$a_k = \begin{cases} \frac{k}{2^k} & \text{if } k \text{ is odd} \\ \frac{1}{3^k} & \text{if } k \text{ is even.} \end{cases}$$

Prove that $\sum_{k=1}^{\infty} a_k$ converges.

For all k, $|a_k| \leq \frac{k}{2^k}$, because

$\frac{1}{3^k} \leq \frac{1}{2^k} \leq \frac{k}{2^k}$.

Since $\lim \frac{k+1}{2^{(k+1)}} \frac{2^k}{k} = \lim \frac{1}{2}\left(1+\frac{1}{k}\right) = \frac{1}{2} < 1$,

$\sum \frac{k}{2^k}$ converges by the ratio test.

So $\sum a_k$ converges by limit comparison.

5. a) (5 points) Prove that $S = \{\frac{1}{2^k} \mid k \in \mathbb{N}\}$ is neither open nor closed.

b) (5 points) Let $T \subset \mathbb{R}$ be a nonempty, open, bounded set. Prove that $\sup(T)$ is **not** an element of T.

a) $\left(\frac{1}{2^k}\right)$ is a sequence in S, but $\frac{1}{2^k} \to 0 \notin S$, so S is not closed.

$\frac{1}{2} = \max S$, but for all $\varepsilon > 0$, $\frac{1}{2} + \frac{\varepsilon}{2} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$, and $\frac{1}{2} + \frac{\varepsilon}{2} > \frac{1}{2}$, so $\frac{1}{2} + \frac{\varepsilon}{2} \notin S$ and $\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right) \not\subseteq S$. So S is not open.

b) If $r \in T$, there exists $\varepsilon > 0$ such that $(r-\varepsilon, r+\varepsilon) \subseteq T$, so $r + \frac{\varepsilon}{2} \in T$, and r is not an upper bound for T. So $\sup(T) \notin T$.

6. (5 points) Prove that $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{N} \\ x^2 & \text{if} \quad x \in \mathbb{R} \backslash \mathbb{N} \end{cases}$$

is not continuous.

For all $n \in \mathbb{N}$, $2 < 2 + \frac{1}{2n} < 3$

so $2 + \frac{1}{2n} \notin \mathbb{N}$.

$2 + \frac{1}{2n} \to 2 + 0 = 2$, but

$f\left(2 + \frac{1}{2n}\right) = \left(2 + \frac{1}{2n}\right)^2 \to 2^2 = 4 \neq f(2) = 2$.