Useful limits (Ross 9.7)

$\frac{1}{n^p} \to 0\ (p>0)$ $\quad a^n \to 0\ (|a|<1)$ $\quad n^{1/n} \to 1$ $\quad a^{1/n} \to 1\ (a>0)$

Pf deferred really well see these

Fact $\sum_{k=m_1}^{\infty} a_k$ converges if and only if $\sum_{k=m_2}$ converges

$$S_n = \sum_{k=m_1}^{n} a_k \qquad t_n = \sum_{k=m_2}^{n} a_k$$

$m_1 \le m_2$ $\quad S_n = a_{m_1} + a_{m_1+1} + \dots + a_{m_2} + \dots + a_n$

$$= \underbrace{(a_{m_1} + \dots + a_{m_2-1})}_{c \text{ constant}} + t_n \qquad S_n = c + t_n$$

Lemma If $|a|<1$, then $\sum a^k = \frac{1}{1-a}$

Proof $\lim_{n\to\infty} \sum_{k=1}^{n} a^k = \lim \frac{1-a^{n+1}}{1-a} = \frac{1-\lim a^{n+1}}{1-a} = \frac{1}{1-a}$ $\quad (a^{n+1} \to 0)$

Proof (Root Test)

Assume $0 \le \limsup |a_k|^{1/k} = v < 1$. Choose $0 < \varepsilon < 1-v$.

Then exists $N \in \mathbb{N}$ such that $(v_N)\ \sup\{|a_k|^{1/k} \mid k>N\} < v+\varepsilon$.

v_N, $v = \limsup$, $v+\varepsilon$, 1

Then for $k>N$, $|a_k|^{1/k} \le v+\varepsilon$ and $|a_k| \le (v+\varepsilon)^k$

Since $\sum_{k=N+1}^{\infty} (v+\varepsilon)^k$ converges ($|v+\varepsilon|<1$, Lemma)

$\sum_{k=N+1}^{\infty} |a_k|$ conv, and $\sum |a_n|$ converges.

Assume $v>1$. then $\{n \mid |a_n|^{1/n} > 1\} = \{n \mid |a_n| > 1\}$ is infinite.

So $\limsup |a_n| \ge 1$ and $a_n \not\to 0$ $\quad$ 1 $\quad v$ $\quad \blacksquare$

Proof (Ratio Test) $\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$

Ex $a_k = \frac{1}{k^k}$ $\quad 0 \le \left|\frac{a_{k+1}}{a_k}\right| = \frac{k^k}{(k+1)^{k+1}} = \frac{1}{k+1}\left(\frac{k}{k+1}\right)^k \le \frac{1}{k+1}$

$|a_k|^{1/k} = \frac{1}{k} \to 0$

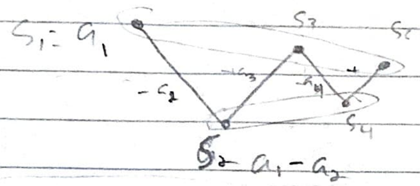
$k \ge 2$ $\quad \frac{1}{k^k} \le \frac{1}{k^2}$, $\sum_{k=2}^{\infty} \frac{1}{k^2}$ conv, so $\sum_{k=2}^{\infty} \frac{1}{k^k}$ conv.

Alternating Series Test

If $a_k \in \mathbb{R}$, $a_k \geq a_{k+1} \geq 0$ for all $k \in \mathbb{N}$, $a_n \to 0$ $(a_1 \geq a_2 \geq a_3 \dots \geq 0)$

Then 1) $\sum (-1)^{k+1} a_k$ converges to $S \in \mathbb{R}$

2) $|S - S_n| \leq a_n$ $\quad \left(S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k \right)$



Proof We want to show (S_n) converges

Fact 1 (S_{2n}) is non decreasing: $S_{2(n+1)} - S_{2n} = -a_{2n+2} + a_{2n+1} \geq 0$

Fact 2 (S_{2n+1}) is non increasing: $S_{2n+1} - S_{2n-1} = a_{2n+1} - a_{2n} \leq 0$

Fact 3 $S_{2n} \leq S_{2m+1}$ for all n, m

If $n \leq m$ $\quad S_{2n} \leq S_{2m} \leq S_{2m+1}$ $\quad (S_{2m+1} - S_{2m} = a_{2m+1})$

If $n > m$, $\quad S_{2n} \leq S_{2n+1} \leq S_{2m+1}$

Thus (S_{2n}) is bounded above, (S_{2n+1}) is bounded below and both converge.

$\lim S_{2m+1} - \lim S_{2m} = \lim (S_{2m+1} - S_{2m}) = \lim a_{2m+1} = 0$

Let $S = \lim S_{2n+1} = \lim S_{2m}$. Then $S_{2n} \leq S \leq S_{2n+1}$ for all n

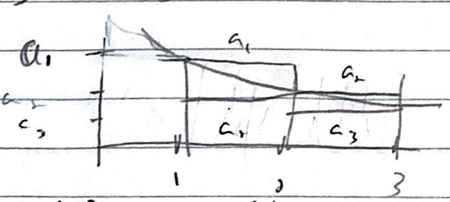
so $|S - S_{2n}| = S - S_{2n} \leq S_{2n+1} - S_{2n} = a_{2n+1} \leq a_{2n}$

$|S - S_{2n+1}| = S_{2n+1} - S \leq S_{2n+1} - S_{2n} = a_{2n+1}$ $\blacksquare$

Integral Test

If $f: \mathbb{R} \to \mathbb{R}$ $\quad x \geq y \Rightarrow f(x) \leq f(y)$ (non increasing), $a_k = f(k)$ $k \in \mathbb{N}$

$\sum a_k$ converges if and only if $\int_1 f(x)dx$ converges



$$\int_1^\infty x^{-p}\,dx = \lim_{t\to\infty} \begin{cases} \dfrac{x^{1-p}}{1-p} - \dfrac{1}{1-p} = \begin{cases} \infty & 1 > p \\ \frac{1}{p-1} & p > 1 \end{cases} \\ \ln(t) \quad p = 1 \quad \infty \end{cases} \Rightarrow \sum \frac{1}{k^p} \text{ conv. iff } p > 1.$$