

Sequentially
compact

All norm spaces
 $\Rightarrow$
 $\Leftarrow$
 $\mathbb{R}^k$

closed and bounded

Theorem 1 Let S be a metric space with distance function d .
If $E \subseteq S$ is sequentially compact, it is closed and bounded.

Theorem 2 Let $E \subseteq \mathbb{R}^k$. If E is closed and bounded, it is sequentially compact.

Proof (Theorem 2)

Let $E \subseteq \mathbb{R}^k$ be closed and bounded.

Let (s_n) be a sequence in E . Then (s_n) is bounded.

By Bolzano-Weierstrass, (s_n) has a convergent subsequence (s_{n_k}) , $k \rightarrow \infty$.
Since (s_{n_k}) is in E and E is closed, $t \in E$.

Recall $E \subseteq S$ is sequentially compact if every sequence (s_n) in E has a subsequence converging to an element of E .

Proof (Theorem 1)

Step 1 A sequentially compact set is closed.

Let E be sequentially compact. Let (s_n) be a sequence in E such that $s_n \rightarrow s$. Then (s_n) has a subsequence which converges to an element of E . But our subsequence converges to s , so $s \in E$ and E is closed.

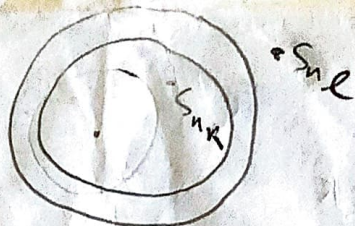
Step 2 A set which is not bounded is not sequentially compact.

Let E be a set which is not bounded. Let $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $E \not\subseteq \{s \in S \mid d(s, t) < n\}$ so there exists $s_n \in \{s \in S \mid d(s, t) \geq n\}$. No subsequence of (s_n) is Cauchy.

Let $t_{n_k} = s_{n_k}$, $n_k < n_{k+1}$, $k = 1, 2, \dots$. Let $N \in \mathbb{N}$ (here

$$\begin{aligned} N > N_1, \quad \ell > \max\{d(s_{n_k}, t) + 1, N\}. \quad \text{Then} \\ d(t_{n_k}, t_{n_\ell}) = d(s_{n_k}, s_{n_\ell}) &\geq d(s_{n_k}, t) - d(s_{n_\ell}, t) \\ &\geq n_k - d(s_{n_\ell}, t) \\ &\geq \ell - d(s_{n_\ell}, t) > 1. \end{aligned}$$

So (t_{n_k}) is not Cauchy, fails with $\epsilon = 1$.



Recall $z \in S_n$ (closed) if and only if for $\epsilon_n (S_n) \neq \emptyset$
 $S_n \rightarrow S$ implies that $S \in \mathbb{C}$

Recall $S \in \mathbb{C}$ is bounded is the ϵ_n for S , $\forall \epsilon \in \mathbb{R}$ $S_n \in \mathbb{C}$
 $\in \{S \in \mathbb{C} \mid d(S, \mathbb{C}) < \epsilon\}$

Useful Limits (Recall from A.7)

$$\frac{1}{n} \rightarrow 0 \quad p > 0$$

$$a^n \rightarrow 0 \quad |a| < 1$$

$$n^{1/n} \rightarrow 1$$

$$a^{1/n} \rightarrow 1 \quad a > 0$$

Series

For $a_1, a_2, \dots, a_n, \dots \in \mathbb{R}$, the symbol $\sum a_k$ (or $\sum_{k=m}^n a_k$)
represents the sequence (S_n) , $S_n = \sum_{k=1}^n a_k$ (or $\sum_{k=m}^n a_k$).

If $S_n \rightarrow S$, we write $\sum a_k = S$ (or $\sum_{k=m}^n a_k = S$).

Defn $\sum a_n$ is Cauchy (i.e. (S_n) is Cauchy) if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that
if $m \geq n > N$, then $|\sum_{k=n}^m a_k| < \epsilon$.

Thm $\sum a_n$ converges if and only if it is Cauchy.

Corollary If $\sum a_n$ converges, $a_n \rightarrow 0$

Proof Given $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for $n \geq N > N$, $|\sum_{k=n}^n a_k| = |a_n| < \epsilon$.

Convergent Series $\rightarrow$ Convergent "Tests" for Series.

Limit Comparison Test

If $|b_n| \leq a_n$ and $\sum a_n$ converges, $\sum b_n$ converges.

Proof

$$\text{By } \Delta, \quad \left| \sum_{k=m}^n b_k \right| \leq \sum_{k=m}^n |b_k| \leq \sum_{k=m}^n a_k = \left| \sum_{k=m}^n a_k \right| \quad (a_k \geq 0)$$

So if $\sum a_k$ is convergent, $\sum b_k$ is convergent. $\square$

Corollary If $0 \leq M_k$, $\sum |b_k|$ does not converge, and $|b_k| \leq M_k$,
then $\sum a_k$ does not converge.

Definition: $\sum a_k$ converges absolutely if $\sum |a_k|$ converges

Then if $\sum a_k$ converges absolutely, $\sum a_k$ converges.
Proof Limit comparison: $|b_k| \leq |b_k|$

Ratio Test Let $a_n \neq 0$

If $\limsup |a_{n+1}/a_n| < 1$, then $\sum a_n$ converges absolutely

If $\liminf |a_{n+1}/a_n| > 1$ or diverges, then $\sum a_n$ does not converge

Root Test

If $\limsup |a_n|^{1/n} < 1$, then $\sum a_n$ converges absolutely

If $\limsup |a_n|^{1/n} > 1$ or diverges, then $\sum a_n$ does not converge.