All metric spaces

Sequentially compact $\Longrightarrow$ closed and bounded

$\Longleftarrow$ $\mathbb{R}^k$

Theorem 1 Let S be a metric space with distance function d. If $E \subseteq S$ is sequentially compact, it is closed and bounded

Theorem 2 Let $E \subseteq \mathbb{R}^k$. If E is closed and bounded, it is sequentially compact

Proof (Theorem 2)

Let $E \subset \mathbb{R}^k$ be closed and bounded.
Let (s_n) be a sequence in E. Then (s_n) is bounded.
By Bolzano-Weierstrass, (s_n) has a convergent subsequence (t_k), $t_k \to t$.
Since (t_k) is in E and E is closed, $t \in E$.

(**Recall** $E \subseteq S$ is sequentially compact if every sequence in E has a subsequence converging to an element of E.)

Proof (Theorem 1)

Step 1 A sequentially compact set is closed

Let E be sequentially compact. Let (s_n) be a sequence in E such that $s_n \to s$. Then (s_n) has a subsequence which converges to an element of E. But any subsequence converges to s, so $s \in E$ and E is closed

Step 2 A set which is not bounded is not sequentially compact.

Let E be a set which is not bounded. Let $t \in S$. Then for each $n \in \mathbb{N}$, $E \not\subseteq \{s \in S \mid d(s, t) < n\}$. So there exists $s_n \in \{s \in S \mid d(s, t) \geq n\}$. No subsequence of (s_n) is Cauchy.

Let $t_k = s_{n_k}$, $n_1 < n_2 < \dots$. Let $N \in \mathbb{N}$. Choose $k > N$, $l > \max\{d(s_{n_k}, t)+1, N\}$. Then

$$
\begin{aligned}
d(t_k, t_l) = d(s_{n_k}, s_{n_l}) &\geq d(s_{n_l}, t) - d(s_{n_k}, t) \\
&\geq n_l - d(s_{n_k}, t) \\
&\geq l - d(s_{n_k}, t) > 1.
\end{aligned}
$$

So (t_k) is not Cauchy, failing with $\varepsilon = 1$.

<u>Recall</u> $E \subseteq S$ is closed if and only if for every (s_n) in E $s_n \to s$, implies that $s \in E$

<u>Recall</u> $E \subseteq S$ is bounded if there exist $t \in S$, $r \in \mathbb{R}$ such that
$$E \subset \{s \in S \mid d(s,t) < r\}.$$

<u>Useful Limits</u> (Ross Thm 9.7)	
$\frac{1}{n^p} \to 0$	$p > 0$
$a^n \to 0$	$\lvert a \rvert < 1$
$n^{1/n} \to 1$	
$a^{1/n} \to 1$	$a > 0$

<u>Series</u>

For $a_1, a_2, \ldots, a_k, \ldots \in \mathbb{R}$, the symbol $\sum a_k$ (or $\sum_{k=m}^{\infty} a_k$) represents the sequence (s_n), $s_n = \sum_{k=1}^{n} a_k$ $\left(\text{or } \sum_{k=m}^{n} a_k\right)$.

If $s_n \to s$, we write $\sum a_k = s$ $\left(\sum_{k=m}^{\infty} a_k = s\right)$

<u>Fact</u> $\sum a_n$ is Cauchy (i.e. (s_n) is Cauchy) if and only if for all $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that if $n \geq m > N$, then $\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$.

<u>Theorem</u> $\sum a_k$ converges if and only if it is Cauchy.

<u>Corollary</u> If $\sum a_k$ converges, $a_k \to 0$

<u>Proof</u> Given $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that for $n \geq n > N$, $\left| \sum_{k=n}^{n} a_n \right| = |a_n| < \varepsilon$.

Comparison theorems for sequences $\rightsquigarrow$ convergence "tests" for series.

<u>Limit Comparison Test</u>

If $|b_k| \leq a_k$, and $\sum a_k$ converges, $\sum b_k$ converges.

Proof

By $\triangle$, $\left|\sum_{k=m}^{n} b_k\right| \le \sum_{k=m}^{n} |b_k| \le \sum_{k=m}^{n} a_k = \left|\sum_{k=m}^{n} a_k\right|$ ($a_k \ge 0$)

So if $\sum a_n$ is Cauchy, $\sum b_n$ is Cauchy. $\blacksquare$

Corollary: If $0 \le b_n$, $\sum b_n$ does not converge, and $|b_n| \le a_n$, then $\sum a_n$ does not converge.

Definition: $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

Thm If $\sum a_n$ converges absolutely, $\sum a_n$ converges

Proof Limit comparison: $|b_n| \le |b_n|$.

Ratio Test Let $a_n \neq 0$

If $\limsup |a_{n+1}/a_n| < 1$, then $\sum a_n$ converges absolutely

If $\liminf |a_{n+1}/a_n| > 1$ or does not exist, then $\sum a_n$ does not converge

Root Test

If $\limsup |a_n|^{1/n} < 1$ then $\sum a_n$ converges absolutely

If $\limsup |a_n|^{1/n} > 1$ or does not exist, then $\sum a_n$ does not converge.